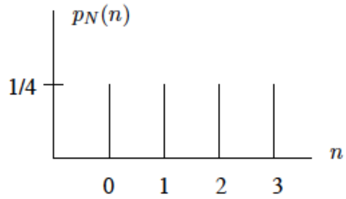
Solutions

1. (Numerical Methods)

2. (Numerical Methods)

3. (a) The first part can be completed without reference to anything other than the die roll:



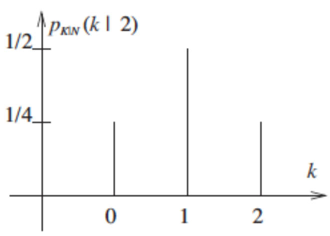
(b) When $N = 0$, the coin is not flipped at all, so $K = 0$. When $N = n$ for $n \in \{1, 2, 3\}$, the coin is flipped n times, resulting in K with a distribution that is conditionally binomial. The binomial probabilities are all multiplied by $1/4$ because $p_N(n) = 1/4$ for $n \in \{0, 1, 2, 3\}$. The joint PMF $p_{N,K}(n, k)$ thus takes the following values and is zero otherwise:

	$k=0$	$k=1$	$k=2$	$k=3$
$n=0$	$1/4$	0	0	0
$n=1$	$1/8$	$1/8$	0	0
$n=2$	$1/16$	$1/8$	$1/16$	0
$n=3$	$1/32$	$3/32$	$3/32$	$1/32$

(c) Conditional on $N = 2, K$ is a binomial random variable. So we immediately see that

$$p_{K|N}(k \mid 2) = \begin{cases} 1/4, \text{ if } k = 0 \\ 1/2, \text{ if } k = 1 \\ 1/4, \text{ if } k = 2 \\ 0, \text{ otherwise} \end{cases}$$

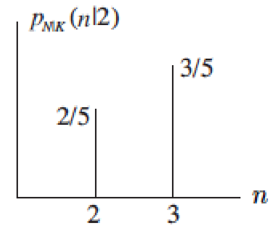
This is a normalized row of the table in the previous part.



(d) To get $K = 2$ heads, there must have been at least 3 coin tosses, so only $N = 3$ and $N = 4$ have positive conditional probability given $K = 2$.

$$p_{N|K}(2 \mid 2) = \frac{\mathbf{P}(\{N = 2\} \cap \{K = 2\})}{\mathbf{P}(\{K = 2\})} = \frac{1/16}{1/16 + 1/32 + 1/32 + 1/32} = 2/5.$$

Similarly, $p_{N|K}(3 \mid 2) = 3/5$.



4. (a) Suppose we choose old widgets. Before we choose any widgets, there are $500 \cdot 0.15 = 75$ defective old widgets. The probability that we choose two defective widgets is

$$\begin{aligned}\mathbf{P}(\text{ two defective} \mid \text{old }) &= \mathbf{P}(\text{ 1st is defective} \mid \text{old }) \cdot \mathbf{P}(\text{ 2nd is defective} \mid \text{1st is defective, old })\\ &= \frac{75}{500}\frac{74}{499} = 0.02224\end{aligned}$$

Now let's consider the new widgets. Before we choose any widgets, there are $1500 \cdot 0.05 = 75$ defective old widgets. Similar to the calculations above,

$$\begin{aligned}\mathbf{P}\ (\text{two defective} \mid \text{new}) &= \mathbf{P}\ (\text{1st is defective} \mid \text{new}) \cdot \mathbf{P}\ (\text{2nd is defective} \mid \text{1st is defective, new})\\ &= \frac{75}{1500}\frac{74}{1499} = 0.002568\end{aligned}$$

By the total probability law,

$$\begin{aligned}\mathbf{P}(\text{ two defective }) =& \mathbf{P}(\text{ old }) \cdot \mathbf{P}(\text{ two defective } \mid \text{ old })\\ &+ \mathbf{P}(\text{ new }) \cdot \mathbf{P}(\text{ two defective } \mid \text{ new })\\ =& \frac{1}{2} \cdot 0.02224 + \frac{1}{2} \cdot 0.002568 = 0.01240\end{aligned}$$

Note that this number is very close to what we would get if we ignored the effects of removing one defective widget before choosing the second widget:

$$\begin{aligned}\mathbf{P}(\text{ two defective }) &= \mathbf{P}(\text{ old }) \cdot \mathbf{P}(\text{ two defective } \mid \text{ old })\\ &+ \mathbf{P}(\text{ new }) \cdot \mathbf{P}(\text{ two defective } \mid \text{ new })\\ &\approx \frac{1}{2} \cdot 0.15^2 + \frac{1}{2} \cdot 0.05^2 = 0.0125\end{aligned}$$

(b) Using Bayes' rule,

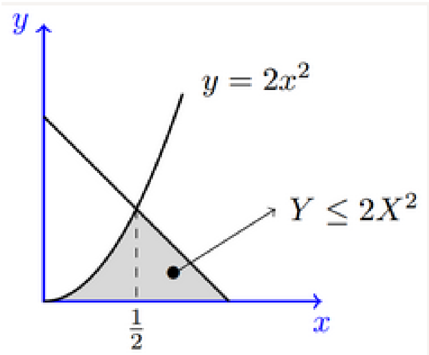
$$\begin{aligned}\mathbf{P}(\text{ old } \mid \text{ two defective }) &= \frac{\mathbf{P}(\text{ old }) \cdot \mathbf{P}(\text{ two defective } \mid \text{ old })}{\mathbf{P}(\text{ old }) \cdot \mathbf{P}(\text{ two defective } \mid \text{ old }) + \mathbf{P}(\text{ new }) \cdot \mathbf{P}(\text{ two defective } \mid \text{ new })}\\ &= \frac{\frac{1}{2} \cdot 0.02224}{\frac{1}{2} \cdot 0.02224 + \frac{1}{2} \cdot 0.002568} = 0.8965\end{aligned}$$

5. 3(a) To find the constant c, we write

$$\begin{aligned}
1 &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)dxdy \\
&= \int_0^1\int_0^{1-x} cx+1dydx \\
&= \int_0^1 (cx+1)(1-x)dx \\
&= \frac{1}{2}+\frac{1}{6}c
\end{aligned}$$

Thus, we conclude $\boldsymbol{c=3}$.

3(b)



To find $P\left(Y<2X^2\right)$, we need to integrate $f_{XY}(x,y)$ over the region shown in Figure. We have

$$\begin{aligned}
P\left(Y<2X^2\right) &= \int_{-\infty}^{\infty}\int_{-\infty}^{2x^2} f_{XY}(x,y)dydx \\
&= \int_0^1\int_0^{\min\left(2x^2,1-x\right)} 3x+1dydx \\
&= \int_0^1 (3x+1)\min\left(2x^2,1-x\right)dx \\
&= \int_0^{\frac{1}{2}} 2x^2(3x+1)dx+\int_{\frac{1}{2}}^1 (3x+1)(1-x)dx \\
&= \frac{53}{96}
\end{aligned}$$

6. (a) Assume that X and Y are independent. Because $p_{X,Y}(3,1)=0$ and $p_Y(1)=1/4, p_X(3)$ must equal zero. This further implies $p_{X,Y}(3,2)=0$ and $p_{X,Y}(3,3)=0$. All the remaining probability mass must go to $(X,Y)=(2,2)$, making $p_{X,Y}(2,2)=5/12, p_X(2)=8/12$, and $p_Y(2)=7/12$. However, $p_{X,Y}(2,2)\neq p_X(2)\cdot p_Y(2)$, contradicting the assumption; thus X and Y are not independent. A simpler explanation uses only two X values and two Y values for which all four (X,Y) pairs have specified probabilities. Note that if X and Y are independent, then $p_{X,Y}(1,3)/p_{X,Y}(1,1)$ and $p_{X,Y}(2,3)/p_{X,Y}(2,1)$ must be equal because they must both equal $p_Y(3)/p_Y(1)$. This necessary equality does not hold, so X and Y are not independent.

(b) Knowing that X and Y are conditionally independent given B, we must have

$$\frac{p_{X,Y}(1,1)}{p_{X,Y}(1,2)}=\frac{p_{X,Y}(2,1)}{p_{X,Y}(2,2)}$$

since the (X, Y) pairs in the equality are all in B. Thus

$$p_{X,Y}(2,2) = \frac{p_{X,Y}(1,2)p_{X,Y}(2,1)}{p_{X,Y}(1,1)} = \frac{(2/12)(2/12)}{1/12} = \frac{4}{12} = \frac{1}{3}$$

(c) Since $\mathbf{P}(B) = 9/12 = 3/4$, we normalize to obtain $p_{X,Y|B}(2,2) = \frac{p_{X,Y}(2,2)}{\mathbf{P}(B)} = 4/9$.

7. (a)

$$X \sim \mathrm{U}(0,1)$$
$$f_X(x) = 1$$

To find the distribution of $-2\log X = Y$ (say) First of all let's settle out domain.

$$0 \leq X \leq 1$$

Taking log

$$-\infty < \log X \leq 0$$

Multiplying -2 we have

$$0 \leq -2\log X < \infty$$
$$0 \leq Y < \infty$$

To find the distribution, we begin with Cumulative distribution function of Y

$$\begin{aligned}F_Y(y) &= P(Y \leq y) = P(-2\log X \leq y) = P\left(\log X \geq \frac{y}{-2}\right)\\ &= 1 - P\left(\log X < \frac{y}{-2}\right)\\ &= 1 - P\left(X < e^{\frac{-y}{2}}\right)\\ &= 1 - F_X\left(e^{\frac{-y}{2}}\right)\end{aligned}$$

Taking the derivative of both sides with respect to variable y we have:

$$f_Y(y) = -f_X\left(e^{\frac{-y}{2}}\right)\left(e^{\frac{-y}{2}}\right)\left(\frac{-1}{2}\right)$$
$$f_Y(y) = \frac{1}{2}e^{\frac{-y}{2}} \quad y \in [0,\infty)$$

(b) Since $V(x) = E\left(x^2\right) - [E(x)]^2 = 1 - 1 = 0$, then the distribution X can only contain the value 1. Meaning, the entire probability is at one point (X = 1). So, $P(X = 1) = 1$ and everywhere else it is 0.

8. (a)

$E(X) = \mu = 20,$ so $\sigma = \sqrt{20} = 4.472$. Therefore, $P(\mu - 2\sigma < X < \mu + 2\sigma) =$ $P(20 - 8.944 < X < 20 + 8.944) \quad = P(11.056 < X < 28.944) = P(X \leq 28) - P(X \leq 11) =$ $F(28;20) - F(11;20) = .966 - .021 = .945$.

(b) The probability that 2 heads appear is $\frac{1}{4}$, that 2 tails (no heads) appear is $\frac{1}{4}$ and that 1 head appears is $\frac{1}{2}$. Thus the probability of winning Rs 200 is $\frac{1}{4}$, of winning Rs 100 is $\frac{1}{2}$, and of losing Rs 500 is $\frac{1}{4}$. Hence $E = 200 \cdot \frac{1}{4} + 100 \cdot \frac{1}{2} - 500 \cdot \frac{1}{4} = -\frac{1}{4} = -25$. That is, the expected value of the game is minus Rs 25 , and so is unfavorable to the player.

9. (a) Let X represent the annual rainfall where X follows a normal distribution with a mean of 40 and a standard deviation of 4. To find $P(X > 50)$, first compute the z-score. The z-score can be found as follows:

$$\begin{aligned} z &= \frac{x-\mu}{\sigma} \\ &= \frac{50-40}{4} \\ &= 2.5 \end{aligned}$$

Now observe the following:

$$\begin{aligned} P(X > 50) &= 1 - P(Z \leq 2.5) \\ &= 1 - 0.9937903 \\ &\approx 0.00621 \end{aligned}$$

Now to find the probability that in 2 of the next 4 years the rainfall will exceed 50 inches, use the binomial distribution where $\boldsymbol{p} = 0.00621$ and $\boldsymbol{q} = 1-p$, let Y be a random variable that represents years where the rainfall exceeds 50 inches and has $p = 0.00621$.

Using the probability mass function for the binomial distribution, we get the following:

$$\begin{aligned} P(Y=2) &= \binom{4}{2}(p)^2 * (1-p)^2 \\ P(Y=2) &= \binom{4}{2}(0.00621)^2 * (1-0.00621)^2 \\ &\approx 0.00023 \end{aligned}$$

So the probability that in 2 of the next 4 years that the region will receive more than 50 inches of annual rainfall is approximately 0.00023.

(b) It is clear that a necessary and sufficient condition for the three segments to form a triangle is that the length of any one of the segments be less than the sum of the other two. Let x, y be the abscissas of the two points chosen at random. Then we must have either

$$0 < x < \frac{1}{2} < y < 1 \text{ and } y - x < \frac{1}{2}$$

or

$$0 < y < \frac{1}{2} < x < 1 \text{ and } x - y < \frac{1}{2}.$$

This is precisely the shaded area in the Figure (See class notes). It follows that the required probability is $\frac{1}{4}$.

10. (a) Here, note that

$$R_{XY} = G = \{(x,y) | x, y \in \mathbb{Z}, |x| + |y| \leq 2\}.$$

Thus, the joint PMF is given by

$$P_{XY}(x,y) = \begin{cases} \frac{1}{13} & (x,y) \in G \\ 0 & \text{otherwise} \end{cases}$$

To find the marginal PMF of $\boldsymbol{X}, P_X(i)$, we use Equation 5.1. Thus,

$$\begin{aligned} P_X(-2) =& P_{XY}(-2,0) = \frac{1}{13} \\ P_X(-1) =& P_{XY}(-1,-1) + P_{XY}(-1,0) + P_{XY}(-1,1) = \frac{3}{13} \\ P_X(0) =& P_{XY}(0,-2) + P_{XY}(0,-1) + P_{XY}(0,0) \\ &+ P_{XY}(0,1) + P_{XY}(0,2) = \frac{5}{13} \\ P_X(1) =& P_{XY}(1,-1) + P_{XY}(1,0) + P_{XY}(1,-1) = \frac{3}{13} \\ P_X(2) =& P_{XY}(2,0) = \frac{1}{13} \end{aligned}$$

Similarly, we can find

$$P_Y(j) = \begin{cases} \frac{1}{13} & \text{for } j = 2, -2 \\ \frac{3}{13} & \text{for } j = -1, 1 \\ \frac{5}{13} & \text{for } j = 0 \\ 0 & \text{otherwise} \end{cases}$$

We can write this in a more compact form as

$$P_X(k) = P_Y(k) = \frac{5 - 2|k|}{13}, \qquad \text{for } k = -2, -1, 0, 1, 2.$$

(b) For $i = -1, 0, 1$, we can write

$$\begin{aligned} P_{X|Y}(i \mid 1) &= \frac{P_{XY}(i,1)}{P_Y(1)} \\ &= \frac{\frac{1}{13}}{\frac{3}{13}} = \frac{1}{3}, \qquad \text{for } i = -1, 0, 1 \end{aligned}$$

Thus, we conclude

$$P_{X|Y}(i \mid 1) = \begin{cases} \frac{1}{3} & \text{for } i = -1, 0, 1 \\ 0 & \text{otherwise} \end{cases}$$

By looking at the above conditional PMF, we conclude that, given $Y = 1, X$ is uniformly distributed over the set $\{-1, 0, 1\}$.

(c) X and Y are not independent. We can see this as the conditional PMF of X given $Y = 1$ (calculated above) is not the same as marginal PMF of $X, P_X(x)$.