30. If $f : \mathbb{R} \to \mathbb{R}$ be continuous, then

$$A := \left\{(x, y) \in \mathbb{R}^2 : yf(x) > 0\right\} \subset \mathbb{R}^2$$

is open.

31. If $f : X \to Y$ maps compact sets to compact sets, then f is continuous.

32. A continuous function $f : \mathbb{R} \to \mathbb{R}$ must be either open or closed.

33. In each of the following situations, give an example of a function $f : S \to T$ such that it is continuous and onto, or else explain why there cannot be such an f : (a) $S = (0, 1), T = (0, 1]$ (b) $S = [0, 1] \cup [2, 3], T = \{0, 1\}$ (c) $S = [0, 1] \times [0, 1], T = \mathbb{R}^2$

34. Let f, g be real valued functions on $[0, 1]$ defined by

$$f(x) := \begin{cases} 0 & \text{if } x \notin \mathbb{Q} \\ 1 & \text{if } x \in \mathbb{Q}; \end{cases} \qquad g(x) := \begin{cases} 0 & \text{if } x \notin \mathbb{Q} \\ x & \text{if } x \in \mathbb{Q}. \end{cases}$$

(a) Show that f is not continuous anywhere in $[0, 1]$.

(b) Show that g is continuous only at $x = 0$.

(c) If $h : [0, 1] \to \mathbb{R}$ is such that $h(x) = c =$ constant for all $x \in \mathbb{Q}$, give a necessary and sufficient condition for h to be continuous on $[0, 1]$. Justify your answer.

35. Let (X, d) be a metric space. Equip $X \times X$ with metric $D : (X \times X) \times (X \times X) \to \mathbb{R}$ defined by

$$D\left((x, y), (x', y')\right) := \max\left\{d\left(x, x'\right), d\left(y, y'\right)\right\}.$$

(a) Show that the metric d, when considered as a real-valued function $d : X \times X \to \mathbb{R}$ on the metric space $X \times X$, is continuous.

(b) Let $f : X \to X$ be a continuous function. Show that $G := \{(x, f(x)) : x \in X\} \subset X \times X$ is closed.

36. (T/F) (a) Let $\pi : \mathbb{R}^2 \to \mathbb{R}$ be the natural projection of $\mathbb{R}^2$ onto the first coordinate, that is, $\pi(x, y) := x$ for any $(x, y) \in \mathbb{R}^2$.

(i) Determine whether π is open on $\mathbb{R}^2$.

(ii) Determine whether π is closed on $\mathbb{R}^2$.

(T/F) (b) If $f : X \to Y$ is open and continuous, must it be closed?

(T/F) (c) If $f : X \to Y$ is closed and continuous, must it be open?

(T/F) (d) If $f : X \to Y$ is open and closed, must it be continuous?

37. For each of the following statements, determine whether it is true or false.

(a) The spaces $(0, 1)$ and $\mathbb{R}$ with Euclidean metric are homeomorphic.

(b) The unit circle $S^1 \subset \mathbb{R}^2$ with Euclidean metric is homeomorphic to $\mathbb{R}$ with Euclidean metric.

(c) The spaces $(0, 1) \cup \{2\}$ and $(0, 1]$ with Euclidean metric are homeomorphic.

38. Show the following statements:

(a) $X := [0, 1] \cup [2, 3]$ is disconnected.

(b) $X := [0, 1) \cup (1, 2]$ is disconnected.

(c) $X := \mathbb{R}\backslash\{0\}$ is disconnected.

(d) $\mathbb{Q}$ is disconnected.

(e) Every discrete metric space with more than one element is disconnected.

(f) Every metric space contains connected subsets.

(g) $S^1\backslash\left\{ \text{ two distinct points on } S^1\right\} \subset \mathbb{R}^2$ is disconnected.

(h) $S^2\backslash S^1 \subset \mathbb{R}^3$ is disconnected, here by S^1 we mean any great circle (i.e., circle of unit length) on the unit sphere S^2.

39. (T/F) (a) If $S, T \subset X$ are disjoint, then $S \cup T$ is disconnected.
    (T/F) (b) If $S \subset X$ is connected, then $S \subset S'$.
    (T/F) (c) If $S \subset X$ is connected, then ∂S is connected.
    (T/F) (d) If $S \subset X$ and ∂S is connected, then S is connected.
    (T/F) (e) Every open ball $B(a, r) \subset X$ is connected.
40. (a) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) := x^2$ is uniformly continuous on $(0, 1]$.
    (b) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) := x^2$ is not uniformly continuous on $(0, \infty)$.
41. State and prove Heine-Borel theorem.
42. State and prove contraction mapping theorem.
43. State and prove Weierstrass approximation theorem.
44. Problems given in lecture notes for Riemann-Stieltjes.
45. All problems discussed in the class room.