Solutions

1.

2.

3. (a) A subset Y of a metric space (X, d) is said to be ***dense*** in X if $\bar{Y} = X$, equivalently if $Y \cap U \neq \emptyset$ for every nonempty open set $U \subset X$.

(b) A subset $Y \subset \mathbb{R}^n$ is ***nowhere dense*** in $\mathbb{R}^n$ if for any nonempty open set $U \subset \mathbb{R}^n$, there is a nonempty open set $V \subset U$ such that $V \cap Y = \emptyset$.

(c) A metric space X is said to be ***disconnected (or separated)*** if $X = A \cup B$, where $A \neq \phi, B \neq \phi, A \cap B = \phi$, and both A, B are open in X. In this case, A and B is called ***separated sets*** of X. X is said to be ***connected*** if it is not disconnected. Equivalently X is connected if it cannot be expressed as the union of two non-empty separated sets.

(d) A metric space (X, d) is ***complete*** if every Cauchy sequence in X is convergent in X.

(e) Let (X, d) be a metric space. A mapping $T : X \to X$ is a ***contraction mapping***, or ***contraction***, if there exists a constant c (called a ***contraction constant***), with $0 \leq c < 1$, such that $d(T(x), T(y)) \leq cd(x, y)$ for all $x, y \in X$.

(f) Let (X, d) be a metric space and $A \subset X$. We say that A is ***compact*** if every open cover $\mathcal{U}$ of A in X has a finite subcover, i.e., if $\mathcal{U} = \{U_j : j \in J\}$ is a collection of open subsets of X with $A \subset \bigcup_{j \in J} U_j$, then there is a finite subcollection $\{U_{j_k} : 1 \leq k \leq p\}$ of $\mathcal{U}$ such that $A \subset \bigcup_{k=1}^{p} U_{j_k}$.

(g) Let α be a monotonically increasing function on $[a, b]$ (since $\alpha(a)$ and $\alpha(b)$ are finite, it follows that α is bounded on [a, b]). Corresponding to each partition P of $[a, b]$, for any real function f which is bounded on [a,b] we write

$$L(f, P, \alpha) := \sum_{j=1}^{k} \inf \{f(x) : \alpha(x_{j-1}) \leq x \leq \alpha(x_j)\} \times (\alpha(x_j) - \alpha(x_{j-1}))$$

$$U(f, P, \alpha) := \sum_{j=1}^{k} \sup \{f(x) : \alpha(x_{j-1}) \leq x \leq \alpha(x_j)\} \times (\alpha(x_j) - \alpha(x_{j-1}))$$

Now, we define

$$L(f, \alpha) = \sup\{L(f, P, \alpha) : P \text{ is a partition of } [a, b]\}$$

$$U(f, \alpha) = \inf\{U(f, P, \alpha) : P \text{ is a partition of } [a, b]\}$$

If $L(f, \alpha) = U(f, \alpha)$, then we say f is integrable with respect to α, in Riemann sense and write $\int_a^b f(x) d\alpha(x)$. This is the Riemann-Stieltjes integral of f with respect to α.

(h) f is ***continuous*** at $x \in X$ if for every $\varepsilon > 0$, there is $\delta > 0$ such that $f(B(x, \delta)) \subset B(f(x), \varepsilon)$; and f is a continuous function if it is continuous at each point of X.

(i) f is ***Lipschitz continuous*** if there is $\lambda > 0$ such that $d'(f(a), f(b)) \leq \lambda d(a, b)$ for every $a, b \in X$.

(j) f is ***uniformly continuous*** if for every $\varepsilon > 0$ there is $\delta > 0$ such that $f(B(x, \delta)) \subset B(f(x), \varepsilon)$ for every $x \in X$.

4. (i) $A^{\circ}=\phi$ as A is not a neighbourhood of any of its points.
(ii) $A^{\prime}=A$ as if we draw an open disc containing (x, y), such that $x+y \in \mathbb{N}$ contains infinitely many points of A.
(iii) $\bar{A}=A \cup A^{\prime}=A$
(iv) iso $(A)=A-A^{\prime}=\phi$
(v) As $A^{\circ} \neq A$, therefore A is not open.
(vi) As $A^{\prime} \subseteq A$, therefore A is closed.
(vii) Clearly, A is not bounded.
(viii) Since A is not bounded, therefore A is not compact.
(ix) As $\bar{A} \neq \mathbb{R}^{2}$, therefore A is not dense.
(x) Since $L_{n}=\{(x, y): x+y=n\}$, then $A=L_{1} \cup\left(\bigcup_{n=2}^{\infty} L_{n}\right), L_{1}$ and $\bigcup_{n=2}^{\infty} L_{n}$ are closed disjoint sets. Therefore, A is not connected.
(xi) As $A^{\prime}=A$, therefore A is perfect.
(xii) As iso $(A) \neq A$, therefore A is not discrete.

5. (a) Suppose that $p \in X$ and $f(p) \in V$. Since V is open, there exists $\varepsilon>0$ such that $y=f(x) \in V$ if $d_{Y}(f(x), f(p))<\varepsilon$, and since f is continuous at p, there exists $\delta>0$ such that $d_{Y}(f(x), f(p))<\varepsilon$ if $d_{X}(x, p)<\delta$. Thus $x \in f^{-1}(V)$ as soon as $d_{X}(x, p)<\delta$.

   (b) If $x, y \in \mathbb{R}^{n}$, then $\left|x_{j}-y_{j}\right| \leq\|x-y\|$, and hence the projection $\pi_{j}: \mathbb{R}^{n} \rightarrow \mathbb{R}$ to the j th coordinate is (Lipschitz) continuous. Any linear map $f: \mathbb{R}^{n} \rightarrow \mathbb{R}^{m}$ may be written as $f=\left(f_{1}, \ldots, f_{m}\right)$, where $f_{j}: \mathbb{R}^{n} \rightarrow \mathbb{R}$ is linear. Note that each f_{j} is a linear combination of projections, and hence continuous. Thus every linear map $f: \mathbb{R}^{n} \rightarrow \mathbb{R}^{m}$ is continuous. We may identify an $m \times n$ real matrix with an element of $\mathbb{R}^{m \times n}=\mathbb{R}^{m n}$. With this identification, we may see that the determinant function det : $\mathbb{R}^{n \times n} \rightarrow \mathbb{R}$ is a finite linear combination of product of projections and hence continuous.

   The preimage of an open set is open from 5(a). Then because a matrix is non-singular if and only if its determinant is not 0 , we have that

$$
\mathrm{GL}_{n}(\mathbb{R})=\left\{A \in M_{n}(\mathbb{R}) \mid A \text { is non-singular }\right\}=\operatorname{det}^{-1}(\mathbb{R}-\{0\})
$$

   and $\mathbb{R}-\{0\}$ is an open set of $\mathbb{R}$, hence $\mathrm{GL}_{n}(\mathbb{R})$ is an open subset of $M_{n}(\mathbb{R})$.

   If $A \in \mathbb{R}^{n \times n}$, then the characteristic polynomial p of A is defined as $p(x)=\operatorname{det}(x I-A)$, and $\operatorname{deg}(p)=n \geq 1$. Since p has at most n roots in $\mathbb{R}$, for any $\varepsilon>0$ we may choose $x \in(-\varepsilon, \varepsilon)$ with $p(x) \neq 0$ and then $\operatorname{det}(A-x I)=(-1)^{n} p(x) \neq 0$. This shows that $\left\{A \in \mathbb{R}^{n \times n}: \operatorname{det}(A) \neq 0\right\}$ is also dense in $\mathbb{R}^{n \times n}$.

6. (a) For any $\left(a_{1}, a_{2}\right) \in B_{2}(0,1)$, we have

$$
\left|a_{1}\right|+\left|a_{2}\right|=d_{2}\left(\left(a_{1}, a_{2}\right),(0,0)\right)<1 .
$$

   It follows that $\left|a_{1}\right|,\left|a_{2}\right|<1$. Hence $d_{1}\left(\left(a_{1}, a_{2}\right),(0,0)\right)=\max \left\{\left|a_{1}\right|,\left|a_{2}\right|\right\}<1$ and so $\left(a_{1}, a_{2}\right) \in B_{1}(0,1)$.

   (b) Yes. For example, define d_{3} by

$$
d_{3}(x, y):=\frac{4}{3} d_{1}(x, y) .
$$

   It is obvious that d_{3} is also a metric, and

$$
\begin{aligned}
B_{3}(0,1) & =\left\{\left(x_{1}, x_{2}\right): \frac{4}{3} \max \left\{\left|x_{1}\right|,\left|x_{2}\right|\right\}<1\right\} \\
& =\left\{\left(x_{1}, x_{2}\right):\left|x_{1}\right|<\frac{3}{4},\left|x_{2}\right|<\frac{3}{4}\right\}
\end{aligned}
$$

Note that

$$\left(\frac{7}{8},0\right) \in B_2(0,1)\backslash B_3(0,1)$$

and

$$\left(\frac{5}{8},\frac{5}{8}\right) \in B_3(0,1)\backslash B_2(0,1)$$

(c) It is not valid in either metric space. Let $L_1 := \{(1,t) : t \in \mathbb{R}\}$, $L_2 := \{(t, 1-t) : t \in \mathbb{R}\}$, and $b := 0 \in \mathbb{R}^2$. Then in $(\mathbb{R}^2, d_1)$,

$$d_1\left(b, L_1\right) = 1 = d_1((0,0),(1,t))$$

for any $t \in \mathbb{R}$ satisfying $|t| \leq 1$, while in $(\mathbb{R}^2, d_2)$,

$$d_2\left(b, L_2\right) = 1 = d_2((0,0),(u,v))$$

for any $u, v \in \mathbb{R}$ satisfying $u \geq 0, v \geq 0, u + v = 1$. (d) (i) For any $N \in \mathbb{N}$,

$$d_1((-N,1),(N,1)) = 2N \geq N$$

Thus $\mathbb{H} \subset (\mathbb{R}^2, d_1)$ is unbounded.

(ii) Yes, such d_4 exists. Define $d_4(x,y) := \frac{d_1(x,y)}{1+d_1(x,y)}$ for any $x, y \in \mathbb{R}^2$. It is easy to verify that d is a well-defined metric on $\mathbb{R}$. For any $x, y \in \mathbb{H}$, we have $d_4(x,y) \leq 1$. Thus $\mathbb{H}$ is bounded in $(\mathbb{R}^2, d_4)$.

7. It is elementary to verify that d is a well-defined metric on M_{22}. Let $\left\{A_n = \left(a_{ij}^n\right)\right\}_{n\in\mathbb{N}}$ be a Cauchy sequence in (M_{22}, d). By definition, for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\max_{1\leq i,j\leq 2} \left|a_{ij}^n - a_{ij}^m\right| = d\left(A_n, A_m\right) < \varepsilon \quad \text{whenever } n, m \geq N.$$

It follows that for any fixed pair (i,j),

$$\left|a_{ij}^n - a_{ij}^m\right| < \epsilon \quad \text{whenever } n, m \geq N.$$

Hence $\left\{a_{ij}^n\right\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$, which must be convergent. Let $a_{ij} := \lim_{n\to\infty} a_{ij}^n$. Then $A := (a_{ij})$ is the limit of A_n in d. In fact, for any $\varepsilon > 0$, for any $i, j = 1, 2$, there exists $N_{ij} \in \mathbb{N}$ such that

$$\left|a_{ij}^n - a_{ij}\right| < \varepsilon \quad \text{whenever } n \geq N_{ij}.$$

Take $N := \max_{1\leq i,j\leq 2} N_{ij}$. Then

$$d\left(A_n, A\right) = \max_{1\leq i,j\leq 2} \left|a_{ij}^n - a_{ij}\right| < \varepsilon \quad \text{whenever } n \geq N.$$

8. [Contraction Mapping Theorem] A contraction T of a complete metric space X has a unique fixed point p.

Proof: If p and p' are two fixed points, item[**??**] implies $d\left(p, p'\right) \leq \alpha d\left(p, p'\right)$, so so $d\left(p, p'\right) = 0$ and $p = p'$. Hence T has at most one fixed point.

To prove it has one, take any point x in X and consider the sequence of iterates:

$$x, \quad T(x), \quad T(T(x)), \ldots$$

That is, define a sequence $\{p_n\}$ inductively as follows:

$$p_0 = x, \quad p_{n+1} = T\left(p_n\right), \quad n = 0, 1, 2, \ldots$$

We will prove that $\{p_n\}$ converges to a fixed point of T. First we show that $\{p_n\}$ is a Cauchy sequence. From the definition of contraction map, we have

$$d\left(p_{n+1}, p_n\right) = d\left(T\left(p_n\right), T\left(p_{n-1}\right)\right) \leq \alpha d\left(p_n, p_{n-1}\right)$$

so, by induction, we find

$$d\left(p_{n+1}, p_n\right) \leq \alpha^n d\left(p_1, p_0\right) = c\alpha^n$$

where $c = d\left(p_1, p_0\right)$. Using the triangle inequality we find, for $m > n$,

$$d\left(p_m, p_n\right) \leq \sum_{k=n}^{m-1} d\left(p_{k+1}, p_k\right) \leq c\sum_{k=n}^{m-1} \alpha^k = c\frac{\alpha^n - \alpha^m}{1-\alpha} < \frac{c}{1-\alpha}\alpha^n$$

Since $\alpha^n \to 0$ as $n \to \infty$, this inequality shows that $\{p_n\}$ is a Cauchy sequence. But X is complete so there is a point p in X such that $p_n \to p$. By continuity of T,

$$T(p) = T\left(\lim_{n\to\infty} p_n\right) = \lim_{n\to\infty} T\left(p_n\right) = \lim_{n\to\infty} p_{n+1} = p$$

so p is a fixed point of T. This completes the proof.

9. (a)

Since f is a continuous function and α is a non-decreasing on [0,1], hence f is RS-integrable and

$$\int_0^1 f d\alpha = \lim_{\|P\|\to 0} S(P, f, \alpha)$$

Consider a partition P of $[0,1]$ as follows:

$$P = \left\{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{r-1}{n}, \frac{r}{n}, \ldots, \frac{n}{n} = 1\right\}.$$

Here rth sub-interval $I_r = [x_{r-1}, x_r] = \left[\frac{r-1}{n}, \frac{r}{n}\right]$, where $r = 1, 2, \ldots, n$ Suppose that $\xi_r = \frac{r}{n} \in \left[\frac{r-1}{n}, \frac{r}{n}\right]$, for $r = 1, 2, \ldots, n$

$$
\begin{aligned}
\therefore \quad S(P, f, \alpha) &= \sum_{r=1}^{n} f\left(\xi_r\right) \delta \alpha_r = \sum_{r=1}^{n} f\left(\xi_r\right)\left[\alpha\left(x_r\right)-\alpha\left(x_{r-1}\right)\right] \\
&= \sum_{r=1}^{n} f\left(\frac{r}{n}\right)\left[\alpha\left(\frac{r}{n}\right)-\alpha\left(\frac{r-1}{n}\right)\right] \\
&= \sum_{r=1}^{n} \frac{r}{n}\left[\frac{r^2}{n^2}-\frac{(r-1)^2}{n^2}\right], \text{ as } f(x)=x, \alpha(x)=x^2 \\
&= \frac{1}{n^3} \sum_{r=1}^{n}\left(2r^2-1\right) = \frac{2}{n^3} \sum_{r=1}^{n} r^2 - \frac{1}{n^3} \sum_{r=1}^{n} r \\
&= \frac{2}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} - \frac{1}{n^3} \cdot \frac{n(n+1)}{2} \\
&= \frac{1}{6n^2} \cdot \left\{2\left(2n^2+3n+1\right)-3n-3\right\} = \frac{1}{6}\left(4+\frac{1}{2n}-\frac{1}{6n^2}\right)
\end{aligned}
$$

$$
\begin{aligned}
\therefore \text{ By (1),} \quad \int_0^1 f d\alpha &= \lim_{\|P\| 00} \frac{1}{6}\left(4+\frac{1}{2n}-\frac{1}{6n^2}\right) = \lim_{n \to \infty} \frac{1}{6}\left(4+\frac{1}{2n}-\frac{1}{6n^2}\right) \\
&= 2/3
\end{aligned}
$$

(b)

$$
\begin{aligned}
\int_0^{10} f(x) d\alpha(x) &= f(10)\alpha(10) - f(0)\alpha(0) - \int_0^{10} \alpha(x) df(x) \\
&= 10 \times 20 - 0 \times 0 - \int_0^{10}(x+[x]) dx \\
&= 200 - 50 - \int_0^{10} [x] dx = 150 - 45 = 105
\end{aligned}
$$

10. (a) True.

Proof. Since a singleton cannot be split into two nonempty subsets, it is clear that every singleton is connected. In particular, every metric space contains connected subsets.

(b) True.

Proof. For any open set $U \subset \mathbb{R}^2$ and any $x \in \pi(U) \subset \mathbb{R}$, there exists $y \in \mathbb{R}$ such that $(x, y) \in U$. Since $U \subset \mathbb{R}^2$ is open, there exists $r > 0$ such that $B_{\mathbb{R}^2}((x, y), r) \subset U$. Hence $x \in (x-r, x+r) = B_{\mathbb{R}}(x, r) \subset \pi\left(B_{\mathbb{R}^2}((x, y), r)\right) \subset \pi(U)$. Therefore, $\pi(U)$ is open and so π is open.

(c) False.

Example: Let $X := \{0\} \subset \mathbb{R}$ and $Y := \mathbb{R}$ with the usual Euclidean metric. Consider $f(x) := x, x \in X$. Then it is evident that f is closed and continuous. However, as the image of the open set $\{0\} \subset X$ equals $\{0\} \subset Y$ which is not open in Y, f is not open.

(d) False.

Example: Let $X := \mathbb{R}, S := [0,1], T := (1,2)$. Clearly S and T are disjoint but $S \cup T = [0,2)$ is connected.

(e) False.

Example: Let $X := \mathbb{R} \backslash \{0\}$. Then $B(1,2) = (-1,0) \cup (0,3)$ is disconnected.