Hints / Answers / Solutions

1. 1. $A^\circ = \phi, A' = \phi, \bar{A} = F$, iso $(A) = F$, not open, closed, bounded, compact, not dense, not connected, not perfect, discrete. 2. $A^\circ = \phi, A' = \phi, \bar{A} = \mathbb{N}$, iso $(A) = \mathbb{N}$, not open, closed, not bounded, not compact, not dense, not connected, not perfect, discrete. 3. $A^\circ = \phi, A' = \phi, \bar{A} = \mathbb{Z}$, iso $(A) = \mathbb{Z}$, not open, closed, not bounded, not compact, not dense, not connected, not perfect, discrete. 4. $A^\circ = \phi, A' = \mathbb{R}, \bar{A} = \mathbb{R}$, iso $(A) = \phi$, not open, not closed, not bounded, not compact, dense, notconnected, not perfect, not discrete. 5. $A^\circ = \phi, A' = \mathbb{R}, \bar{A} = \mathbb{R}$, iso $(A) = \phi$, not open, not closed, not bounded, not compact, dense, not connected, not perfect, not discrete. 6. $A^\circ = \mathbb{R} \quad \mathbb{N}, A' = \mathbb{R}, \bar{A} = \mathbb{R}$, iso $(A) = \phi$, open, not closed, not bounded, not compact, dense, not connected, not perfect, not discrete. 7. $A^\circ = \mathbb{R} \quad \mathbb{Z}, A' = \mathbb{R}, \bar{A} = \mathbb{R}$, iso $(A) = \phi$, open, not closed, not bounded, not compact, dense, not connected, not perfect, not discrete. 8. $A^\circ = \phi, A' = \mathbb{R}, \bar{A} = \mathbb{R}$, iso $(A) = \phi$, not open, not closed, not bounded, not compact, dense, not connected, not perfect, not discrete. 9. $A^\circ = \phi, A' = \mathbb{R}, \bar{A} = \mathbb{R}$, iso $(A) = \phi$, not open, not closed, not bounded, not compact, dense, not connected, not perfect, not discrete. 10. $A^\circ = (a, b), A' = [a, b], \bar{A} = [a, b]$, iso $(A) = \phi$, open, not closed, bounded, not compact, not dense, connected, not perfect, not discrete. 11. $A^\circ = (a, b), A' = A, \bar{A} = [a, b]$, iso $(A) = \phi$, not open, closed, bounded, compact, not dense, connected, perfect, not discrete. 12. $A^\circ = (a, b), A' = [a, b], \bar{A} = [a, b]$, iso $(A) = \phi$, not open, not closed, bounded, not compact, not dense, connected, not perfect, not discrete. 13. $A^\circ = (a, b), A' = [a, b], \bar{A} = [a, b]$, iso $(A) = \phi$, not open, not closed, bounded, not compact, not dense, connected, not perfect, not discrete. 14. $A^\circ = \phi, A' = \phi, \bar{A} = \phi$, iso $(A) = \phi$, open, closed, bounded, compact, not dense, connected, perfect, discrete. 15. $A^\circ = \mathbb{R}, A' = \mathbb{R}, \bar{A} = \mathbb{R}$, iso $(A) = \phi$, open, closed, not bounded, not compact, dense, connected, perfect, not discrete. 16. $A^\circ = (1, 2), A' = [1, 2], \bar{A} = [1, 2] \cup \{3, 4, 5\}$, iso $(A) = \{3, 4, 5\}$, not open, not closed, bounded, not compact, not dense, not connected, not perfect, not discrete. 17. $A^\circ = \phi, A' = \{0\}, \bar{A} = \ \frac{1}{n} : n \in \mathbb{N}\big) \cup \{0\}$, iso $(A) = \frac{1}{n}$, not open, not closed, bounded, not compact, not dense, not connected, not perfect, discrete. 18. $A^\circ = \phi, A' = \{1, \quad 1\}, \bar{A} = A \cup \{1, \quad 1\}$, iso $(A) = A$, not open, not closed, bounded, not compact, not dense, not connected, not perfect, discrete. 19. $A^\circ = \phi, A' = \phi, \bar{A} = A$, iso $(A) = A$, not open, closed, not bounded, not compact, not dense, not connected, not perfect, discrete. 20. $A^\circ = \phi, A' = \mathbb{N}, \bar{A} = A \cup A'$, iso $(A) = A \quad A'$, not open, not closed, not bounded, not compact, not dense, not connected, not perfect, not discrete. 21. $A^\circ = \phi, A' = \frac{1}{m} \cup \{0\}, \bar{A} = \left\{\frac{1}{m} + \frac{1}{n}\right\} \cup \{0\}$, iso $(A) = A \quad A'$, not open, not closed, bounded, not compact, not dense, not connected, not perfect, not discrete. 22. $A^\circ = \phi, A' = \left\{\pm\frac{1}{n} : n \in \mathbb{N}\right\} \cup \{0\}, \bar{A} = A \cup \{0\}$, iso $(A) = A \quad A'$, not open, not closed, bounded, not compact, not dense, not connected, not perfect, not discrete. 23. $A^\circ = (0, 2), A' = [0, 2], \bar{A} = [0, 2]$, iso $(A) = \phi$, not open, not closed, bounded, not compact, not dense, connected, not perfect, not discrete. 24. $A^\circ = \phi, A' = \{1\}, \bar{A} = \left\{1 + \frac{1}{3^n}\right\} \cup \{1\}$, iso $(A) = 1 + \frac{1}{3^n}$, not open, not closed, bounded, not compact, not dense, not connected, not perfect, discrete. 25. $A^\circ = \phi, A' = \phi, \bar{A} = A, \text{iso}(A) = A$, not open, closed, not bounded, not compact, not dense, not connected, not perfect, discrete. 26. $A^\circ = \phi, A' = \{3^n : n \in \mathbb{N}\}, \bar{A} = A \cup A'$, iso $(A) = A \quad A'$, not open, not closed, not bounded, not compact, not dense, not connected, not perfect, not discrete. 27. $A^\circ = (0, \infty), A' = [0, \infty), \bar{A} = [0, \infty)$, iso $(A) = \phi$, open, not closed, not bounded, not compact, not dense, connected, not perfect, not discrete. 28. $A^\circ = (\quad \infty, \quad 1), A' = (\quad \infty, \quad 1], \bar{A} = (\quad \infty, \quad 1]$, iso $(A) = \phi$, open, not closed, not bounded, not compact, not dense, connected, not perfect, not discrete. 29. $A^\circ = \phi, A' = \{1, \quad 1\}, \bar{A} = \left\{(\quad 1)^n + \frac{1}{n}\right\} \cup \{1, \quad 1\}$, iso $(A) = \left\{(\quad 1)^n + \frac{1}{n}\right\}$, not open, not closed, bounded, not compact, not dense, not connected, not perfect, discrete. 30. $A^\circ = \phi, A' = 0, \bar{A} = A$, iso $(A) = A \quad \{0\}$, not open, closed, bounded, compact, not dense, not connected, not perfect, not discrete. 31. $A^\circ = \phi, A' = \{1\}, \bar{A} = A \cup \{1\}$, iso $(A) = A$, not open, not closed, bounded, not compact, not dense, not connected, not perfect, discrete. 32. $A^\circ = \phi, A' = e, \bar{A} = A \cup \{e\}$, iso $(A) = A$, not open, not closed, bounded, not compact, not dense, not connected, not perfect, discrete. 33. $A^\circ = \phi, A' = \left\{\frac{1}{2^m} : m \in \mathbb{N}\right\} \cup \left\{\frac{1}{3^n} : n \in \mathbb{N}\right\} \cup \{0\}, \bar{A} = A \cup A'$, iso $(A) = A$, not open, not closed, bounded, not compact, not dense, not connected, not perfect, discrete. 34. $A^\circ = \phi, A' = 0, \bar{A} = \left\{\frac{1}{2^n} + \frac{1}{3^n}\right\} \cup \{0\}$, iso $(A) = A$, not open, not closed, bounded, not compact, not dense, not connected, not perfect, discrete. 35. $A^\circ = \phi, A' = \left\{\frac{1}{2^m} : m \in \mathbb{N}\right\} \cup \left\{\frac{1}{3^n} : n \in \mathbb{N}\right\} \cup \left\{\frac{1}{5^r} : r \in \mathbb{N}\right\} \cup \left\{\frac{1}{2^m} + \frac{1}{3^n} : m, n \in \mathbb{N}\right\} \cup \left\{\frac{1}{5^r} + \frac{1}{3^n} : r, n \in \mathbb{N}\right\}, \cup \left\{\frac{1}{2^m} + \frac{1}{5^r} : m, r \in \mathbb{N}\right\} \cup \{0\}, \bar{A} = A \cup A'$, iso $(A) = A$, not open, not closed, bounded, not compact, not dense, not connected, not perfect, discrete. 36. $A^\circ = \phi, A' = \{0\}, \bar{A} = A \cup \{0\}$, iso $(A) = A$, not open, not closed, bounded, not compact, not dense, not connected, not perfect, discrete. 37. $A^\circ = \phi, A' = \{0\}, \bar{A} = A \cup \{0\}$, iso $(A) = A$, not open, not closed, bounded, not compact, not dense, not connected, not perfect, discrete. 38. $A^\circ = \phi, A' = \{1\}, \bar{A} = A \cup \{1\}$, iso $(A) = A$, not open, not closed, bounded, not compact, not dense, not connected, not perfect, discrete. 39. $A^\circ = \phi, A' = \{1\}, \bar{A} = A \cup \{1\}$, iso $(A) = A$, not open, not closed, bounded, not compact, not dense, not connected, not perfect, discrete. 40. $A^\circ = \phi, A' = \left\{\sin \quad \frac{1}{n}\big) + 1 : n \in \mathbb{N}\right\} \cup \left\{\cos \quad \frac{1}{n}\big) : n \in \mathbb{N}\right\} \cup \{1\}, \bar{A} = A \cup A'$, iso $(A) = A \quad A'$, not open, not

closed, bounded, not compact, not dense, not connected, not perfect, not discrete.

2. (i) Finite set (ii) $\left\{\frac{1}{n} : n \in \mathbb{N}\right\}$ (iii) $\{1+(\quad 1)^n : n \in \mathbb{N}\}$ (iv) $[a,b]$ (v) $\left\{5+\frac{1}{n} : n \in \mathbb{N}\right\}$

3. (i) $[a,b]$ (ii) $\mathbb{R} \quad \mathbb{N}$ (iii) (iv) $\phi, \mathbb{R}$

4. **a)** (i) $A^\circ = \{(x,y) : x > 0, y > 0\}$ as there exist an open disc centered at (x_0, y_0) such that $x_0 > 0$ and $y_0 > 0$ which is entirely contained in A. (ii) $A' = \{(x,y) : x \geq 0, y \geq 0\}$ as every open disc containing (x,y) such that $x \geq 0, y \geq 0$, contains infinitely many points of A. (iii) $\bar{A} = A \cup A' = A'$ (iv) iso $(A) = A \quad A' = \phi$ (v) As $A^\circ \neq A$, therefore A is not open. (vi) As $A' \quad A$, therefore A is not closed. (vii) Clearly, A is not bounded. (viii) Since A is neither bounded nor closed, therefore A is not compact. (ix) As $\bar{A} \neq \mathbb{R}^2$, therefore A is not dense. (x) A is path-connected as we can join any two points of the set A by a continuous curve within the set A. (xi) A is connected as it is path-connected. (xii) A is convex as it contains all the points of any line segment joining any two points of the set A. (xiii) As $A' \neq A$, therefore A is not perfect. (xiv) As iso $(A) \neq A$, therefore A is not discrete.

   b) (i) $A^\circ = A$ as there exist an open disc centered at (x_0, y_0) such that $(x_0, y_0) \in A$ which is entirely contained in A. (ii) $A' = \left\{(x,y) : x^2 + y^2 \leq 1\right\} \cup \left\{(x,y) : (x \quad 2)^2 + y^2 \leq 1\right\}$ as every open disc containing (x,y) such that $x^2 + y^2 \leq 1$ or $(x \quad 2)^2 + y^2 \leq 1$, contains infinitely many points of A. (iii) $\bar{A} = A \cup A' = A'$ (iv) iso $(A) = A \quad A' = \phi$ (v) As $A^\circ = A$, therefore A is open. (vi) As $A' \quad A$, therefore A is not closed. (vii) Clearly, A is bounded. (viii) Since A is not closed, therefore A is not compact. (ix) As $\bar{A} \neq \mathbb{R}^2$, therefore A is not dense. (x) A is not path connected as we cannot join some of the points of the set A by a continuous curve within the set A. (xi) A is not connected as it is the union of two non empty disjoint open set. (xii) A is not convex as it does not contain all the points of line segment joining some of the two points of the set A. (xiii) As $A' \neq A$, therefore A is not perfect. (xiv) As iso $(A) \neq A$, therefore A is not discrete.

   c) (i) $A^\circ = \phi$ as A is not a neighbourhood of any of its points. (ii) $A' = \phi$ as if we draw an open disc containing (x,y), contains no points of A other than (x,y). (iii) $\bar{A} = A \cup A' = A$ (iv) iso $(A) = A \quad A' = A$ (v) As $A^\circ \neq A$, therefore A is not open. (vi) As $A' \subseteq A$, therefore A is closed. (vii) Clearly, A is not bounded. (viii) Since A is not bounded, therefore A is not compact. (ix) As $\bar{A} \neq \mathbb{R}^2$, therefore A is not dense. (x) A is not path connected as we cannot join some of the two points of the set A by a continuous curve within the set A. (xi) We know that $\prod_i X_i$ is connected iff each X_i is connected and since $\mathbb{N}$ is not connected, therefore $A = \mathbb{N} \times \mathbb{N}$ is not connected. (xii) A is not convex as it does not contain some of the points of any line segment joining any two points of the set A. (xiii) As $A' \neq A$, therefore A is not perfect. (xiv) As iso $(A) = A$, therefore A is discrete.

   d) Let $L_n = \{(x,y) : x + y = n\}$ (i) $A^\circ = \phi$ as A is not a neighbourhood of any of its points. (ii) $A' = A$ as if we draw an open disc containing (x,y), such that $x + y \in \mathbb{N}$ contains infinitely many points of A. (iii) $\bar{A} = A \cup A' = A$ (iv) iso $(A) = A \quad A' = \phi$ (v) As $A^\circ \neq A$, therefore A is not open. (vi) As $A' \subseteq A$, therefore A is closed. (vii) Clearly, A is not bounded. (viii) Since A is not bounded, therefore A is not compact. (ix) As $\bar{A} \neq \mathbb{R}^2$, therefore A is not dense. (x) A is not path connected as we cannot join some of the points of the set A by a continuous curve within the set A. (xi) Since $L_n = \{(x,y) : x + y = n\}$, then $A = L_1 \cup \left(\bigcup_{n=2}^{\infty} L_n\right), L_1$ and $\bigcup_{n=2}^{\infty} L_n$ are closed disjoint sets. Therefore, A is not connected. (xii) A is not convex as it does not contain all the points of the line segment joining some of the two points of the set A. (xiii) As $A' = A$, therefore A is perfect. (xiv) As iso $(A) \neq A$, therefore A is not discrete.

   e) (i) $A^\circ = \phi$ as A is not a neighbourhood of any of its points. (ii) $A' = \mathbb{R}^2$ as every point of $\mathbb{R}^2$ is a limit point of A. (iii) $\bar{A} = A \cup A' = \mathbb{R}^2$ (iv) iso $(A) = A \quad A' = \phi$ (v) As $A^\circ \neq A$, therefore A is not open. (vi) As $A' \quad A$, therefore A is not closed. (vii) Clearly, A is not bounded. (viii) Since A is not bounded, therefore A is not compact. (ix) As $\bar{A} = \mathbb{R}^2$, therefore A is dense. (x) A is not path connected as we cannot join some of the points of the set A by a continuous curve within the set A. (xi) Let $f : A \to \mathbb{Q}$ be a map such that $f(x,y) = x + y$, then f is continuous and onto and $f(A) = \mathbb{Q}$ and $\mathbb{Q}$ is not connected. So, A is not connected. (xii) A is not convex as it does not contain all the points of the line segment joining some of the two points of the set A. (xiii) As $A' \neq A$, therefore A is not perfect. (xiv) As iso $(A) \neq A$, therefore A is not discrete.

5. Let $a_n = \frac{1}{n^2}$ and let $\varepsilon > 0$ be given. Then for $n > m$, we have

$$|a_n \quad a_m| = \left|\frac{1}{n^2} \quad \frac{1}{m^2}\right| = \frac{1}{m^2} \quad \frac{1}{n^2} < \frac{1}{m^2} < \varepsilon \text{ whenever } m^2 > \frac{1}{\varepsilon} \quad \left[\because \frac{1}{n^2} < \frac{1}{m^2}\right]$$

   i.e. whenever $m > \frac{1}{\sqrt{\varepsilon}}$

   Thus if we choose a positive integer $m > \frac{1}{\sqrt{\varepsilon}}$ then $\quad |a_n \quad a_m| < \varepsilon \forall n \geq m$ Hence $\langle a_n \rangle$ is a Cauchy sequence.

6. Let $a_n = (\ \ 1)^n$. To show that $< a_n >$ is not a Cauchy sequence, we have to find an $\varepsilon_0 > 0$ such that for every positive integer m there exist at least one $n > m$ such that

$$|a_n \quad a_m| > \varepsilon_0$$

Let $\varepsilon_0 = \frac{1}{2}$, then for every integer m, we have an integer $m + 1 > m$ such that

$$|a_{m+1} \quad a_m| = \left|(\ \ 1)^{m+1} \quad (\ \ 1)^m\right| = 2 > \frac{1}{2}$$

Thus, $\langle(\ \ 1)^n\rangle$ is not a Cauchy sequence.

7. Do it yourself.

8. Answer: True. Proof: (M1) and (M2) are clearly satisfied. For any $x, y, z \in \mathbb{R}$,

$$\begin{aligned} d(x,y)^2 &= |x \quad y| \\ &\le |x \quad z| + |z \quad y| \\ &= d(x,z)^2 + d(z,y)^2 \\ &\le [d(x,z) + d(z,y)]^2, \end{aligned}$$

hence (M3) is also satisfied.

9. Answer: False. Example: Take $(x_1, y_1) = (1, 0)$ and $(x_2, y_2) = (0, 1)$. Then

$$\begin{aligned} &d\left((x_1, y_1), (x_2, y_2)\right) \\ &\quad = 4 > 2 = d\left((x_1, y_1), (0,0)\right) + d\left((0,0), (x_2, y_2)\right) \end{aligned}$$

and so the triangle inequality is violated.

10. Answer: False. Example: Take $n = 2, x := (1, 0), y := (\ \ 1, 0)$. Then $x \neq y$ but $d(x, y) = 0$.

11. Answer: False. Example: Let $X := \{p, q, r\}$ and d_1, d_2 be defined by $\begin{array}{c|ccc} d_1 & p & q & r \\ \hline p & 0 & 10 & 1 \\ q & 10 & 0 & 10 \\ r & 1 & 10 & 0 \end{array}$, $\begin{array}{c|ccc} d_2 & p & q & r \\ \hline p & 0 & 10 & 10 \\ q & 10 & 0 & 1 \\ r & 10 & 1 & 0 \end{array}$.

It is easily shown that d_1 and d_2 are well-defined metrics on X. However,

$$\begin{aligned} d(p,q) &= \min\{d_1(p,q), d_2(p,q)\} = \min\{10, 10\} = 10 \\ d(p,r) &= \min\{d_1(p,r), d_2(p,r)\} = \min\{1, 10\} = 1 \\ d(r,q) &= \min\{d_1(r,q), d_2(r,q)\} = \min\{10, 1\} = 1 \end{aligned}$$

and so

$$d(p,q) = 10 \not\leq 2 = 1 + 1 = d(p,r) + d(r,q).$$

12. Answer: False. Example: Let $X := \mathbb{R}, S := (0, 1)$, then $0 \notin S$ but $d(0, S) = \inf\{d(0, s) : s \in S\} = \inf\{s : 0 < s < 1\} = 0$.

13. Answer: False. Example: Let $X := \mathbb{R}, S := (0, 1), T := (1, 2)$. Then $S \cap T = \phi$ but $d(S, T) = \inf\{d(s, t) : s \in S, t \in T\} = 0$.

14. Answer: (a) No. Justification. Since $d\ \ 1, \frac{1}{2}) = \left[\left|1 \quad \frac{1}{2}\right|\right] = \left[\frac{1}{2}\right] = 0$ but $1 \neq \frac{1}{2}$, (M1) is violated. (b) No. Justification. Since

$$d(0,2) = (2 \quad 0)^2 = 4 \not\leq 2 = 1^2 + 1^2 = d(0,1) + d(1,2),$$

(M3) is violated. (c) Yes. Proof. It is clear that $d(x, y) \geq 0$ and $d(x, x) = 0$ for all x, $y \in \mathbb{R}$. Conversely, if $d(x, y) = 0$, then

$$\frac{x}{1+|x|} = f(x) = f(y) = \frac{y}{1+|y|}.$$

Taking absolute values on both sides and simplify, we have $|x| = |y|$. Putting it back to (*), we have $x = y$ and so (M1) is satisfied. (M2) is obvious. (M3) follows immediately from the usual triangle inequality for absolute value:

$$\begin{aligned} d(x,y) &= |f(x) \quad f(y)| \\ &\leq |f(x) \quad f(z)| + |f(z) \quad f(y)| \\ &= d(x,z) + d(z,y). \end{aligned}$$

15. Answer: False. Example: Take $X = \mathbb{R}$ and $d = d'$ to be the Euclidean metric. Then D fails to be a metric as it violates the triangle inequality (M3). In fact, for $x = 1, y = 2$, and $z = 3$, we have

$$D(x,z) = 2 \cdot 2 = 4 > 1 + 1 = D(x,y) + D(y,z).$$

16. (a) False, d is not a metric. Justification. In fact, for $u = \ \begin{smallmatrix}1\\0\end{smallmatrix}\big)$ and $v = \ \begin{smallmatrix}0\\0\end{smallmatrix}\big)$, we have

$$(u \quad v)^T A(u \quad v) = \quad \frac{5}{2} < 0$$

and $d(u,v)$ is not defined and in particular, d is not a metric.

(b) Yes, d is a metric. Proof. The conditions $d(u,v) \geq 0, d(u,u) = 0$ and $d(u,v) = d(v,u)$ follow directly from that d_1 and d_2 are metrics. On the other hand, if $\max\{d_1(u,v), d_2(u,v)\} = d(u,v) = 0$, we must have $d_1(u,v) = d_2(u,v) = 0$ and so $u = v$. Finally, for any $u, v, w \in \mathbb{R}^2$, we have either $d(u,v) = d_1(u,v)$ or $d(u,v) = d_2(u,v)$. Without loss of generality, assume that $d(u,v) = d_1(u,v)$. Then as $d_1 \leq d$, we have

$$\begin{aligned} d(u,v) = d_1(u,v) &\leq d_1(u,w) + d_1(w,v) \\ &\leq d(u,w) + d(w,v) \end{aligned}$$

Hence (M1)-(M3) are satisfied and so d is a metric.

17. (a) For any $(a_1, a_2) \in B_2(0,1)$, we have

$$|a_1| + |a_2| = d_2\left((a_1, a_2), (0,0)\right) < 1.$$

It follows that $|a_1|, |a_2| < 1$. Hence $d_1\left((a_1, a_2), (0,0)\right) = \max\{|a_1|, |a_2|\} < 1$ and so $(a_1, a_2) \in B_1(0,1)$.

(b) Yes. For example, define d_3 by

$$d_3(x,y) := \frac{4}{3} d_1(x,y).$$

It is obvious that d_3 is also a metric, and

$$\begin{aligned} B_3(0,1) &= \left\{(x_1, x_2) : \frac{4}{3}\max\{|x_1|, |x_2|\} < 1\right\} \\ &= \left\{(x_1, x_2) : |x_1| < \frac{3}{4}, |x_2| < \frac{3}{4}\right\} \end{aligned}$$

Note that

$$\left(\frac{7}{8}, 0\right) \in B_2(0,1) \backslash B_3(0,1)$$

and

$$\left(\frac{5}{8}, \frac{5}{8}\right) \in B_3(0,1) \backslash B_2(0,1)$$

(c) It is not valid in either metric space. Let $L_1 := \{(1,t) : t \in \mathbb{R}\}, L_2 := \{(t, 1 \quad t) : t \in \mathbb{R}\}$, and $b := 0 \in \mathbb{R}^2$. Then in $\ \mathbb{R}^2, d_1\big)$,

$$d_1(b, L_1) = 1 = d_1((0,0),(1,t))$$

for any $t \in \mathbb{R}$ satisfying $|t| \leq 1$, while in $\ \mathbb{R}^2, d_2\big)$,

$$d_2(b, L_2) = 1 = d_2((0,0),(u,v))$$

for any $u, v \in \mathbb{R}$ satisfying $u \geq 0, v \geq 0, u+v=1$. (d) (i) For any $N \in \mathbb{N}$,

$$d_1((-N,1),(N,1)) = 2N \geq N$$

Thus $\mathbb{H} \subset \left(\mathbb{R}^2, d_1\right)$ is unbounded.

(ii) Yes, such d_4 exists. Define $d_4(x,y) := \frac{d_1(x,y)}{1+d_1(x,y)}$ for any $x, y \in \mathbb{R}^2$. It is easy to verify that d is a well-defined metric on $\mathbb{R}$. For any $x, y \in \mathbb{H}$, we have $d_4(x,y) \leq 1$. Thus $\mathbb{H}$ is bounded in $\left(\mathbb{R}^2, d_4\right)$.

18. Solution: Recall first that interior points of a set are automatically points in the given set. So to find all interior points of a set, we only need to consider the points in the given set and study whether they are interior points or not.

(a) Answer: $(0,\infty)$ In fact, for any $x \in (0,\infty)$, $B_{\mathbb{R}}\left(x, \frac{x}{2}\right) = \left(\frac{x}{2}, \frac{3x}{2}\right) \subset [0,\infty)$ and so every point in $(0,\infty)$ is an interior point of $[0,\infty)$ in $\mathbb{R}$. It remains to consider the point $0 \in [0,\infty)$. As $B_{\mathbb{R}}(0,r) = (-r,r) \not\subset [0,\infty)$ for any $r > 0$, we see that 0 is not an interior point of $[0,\infty)$ in $\mathbb{R}$. Thus the interior of $[0,\infty)$ in $\mathbb{R}$ is $(0,\infty)$.

(b) Answer: $(0,1)$. In fact, for any $x \in (0,1)$, if we write $r := \min\{x, 1-x\} > 0$, then $B_{\mathbb{R}}(x,r) \subset [0,1)$. Hence every point in $(0,1)$ is an interior point of $[0,1)$ in $\mathbb{R}$. It remains to consider the point $0 \in [0,1)$. As $B_{\mathbb{R}}(0,r) = (-r,r) \not\subset [0,1)$ for any $r > 0$, we see that 0 is not an interior point of $[0,1)$ in $\mathbb{R}$. Thus the interior of $[0,1)$ in $\mathbb{R}$ is $(0,1)$.

(c) Answer: $[0,1)$. Similar to part (b) above, for any $x \in (0,1)$, write $r := \min\{x, 1-x\} > 0$. We have $B_{\mathbb{R}}(x,r) \subset [0,1)$ and so $B_{[0,\infty)}(x,r) = B_{\mathbb{R}}(x,r) \cap [0,\infty) \subset [0,1) \cap [0,\infty) = [0,1)$. So every point in $(0,1)$ is an interior point of $[0,1)$ in $[0,\infty)$. It remains to consider the point $0 \in [0,1)$. As $B_{[0,\infty)}(0,r) = B_{\mathbb{R}}(0,r) \cap [0,\infty) = (-r,r) \cap [0,\infty) = [0,r) \subset [0,1)$ for any $0 < r < 1$, we see that 0 is an also interior point of $[0,1)$ in $[0,\infty)$. Therefore, the whole set $[0,1)$ is the interior of $[0,1)$ in $[0,\infty)$.

(d) Answer: ϕ. In fact, for any $x \in \mathbb{Q}$ and any $r > 0$, $B_{\mathbb{R}}(x,r) = (x-r, x+r)$ is bound to contain irrational numbers. Hence $B_{\mathbb{R}}(x,r) \not\subset \mathbb{Q}$. Therefore, no point in $\mathbb{Q}$ can be an interior point of $\mathbb{Q}$ in $\mathbb{R}$.

19. (a) False. Justification. It is obvious that $B_{\mathbb{R}}\left((-1)^n + \frac{1}{n}, r\right) \not\subset S$ for any $n \in \mathbb{N}$ and any $r > 0$. Hence no point in S is an interior point of S in $\mathbb{R}$ and so S is not open in $\mathbb{R}$.

(b) False. Justification. Consider $1 \in \mathbb{R}\backslash S$. For any $r > 0, 1 + \frac{1}{2r} \in S \cap B_{\mathbb{R}}(1,r)$. So $1 \in \mathbb{R}\backslash S$ is not an interior point of $\mathbb{R}\backslash S$ and so $\mathbb{R}\backslash S$ is not open in $\mathbb{R}$. Hence S is not closed in $\mathbb{R}$.

(c) False. Justification. Similar to (a).

(d) False. Justification. Similar to (b).

(e) True. Proof. Observe first that T contains all the positive elements of S. For any $t \in T$, write $t = 1 + \frac{1}{2n}$, where $n \in \mathbb{N}$. Note that all elements in $B_S\left(t, \frac{1}{2n}\right)$ are positive and so $B_S\left(t, \frac{1}{2n}\right) \subset T$. Hence T is open in S.

20. Answer: True. Proof. Let $S \subset X$ be a finite set. Write $S = \{x_1, \ldots, x_n\}$. If $\{U_\alpha\}_{\alpha \in \Lambda}$ is an open cover of S in X, then for any $i = 1, \ldots, n$, there is an $\alpha_i \in \Lambda$ such that $x_i \in U_{\alpha_i}$ and so $\{U_{\alpha_i} : i = 1, \ldots, n\}$ is a finite subcover of $\{U_\alpha\}_{\alpha \in \Lambda}$.

21. Answer: False. Justification. Since $\mathbb{N}$ is unbounded. By Theorem 1.3.5, $\mathbb{N}$ is not compact. Alternatively, $\left\{\left(n - \frac{1}{2}, n + \frac{1}{2}\right) : n \in \mathbb{N}\right\}$ is an open cover of $\mathbb{N}$ without finite subcover. Hence $\mathbb{N}$ is not compact.

22. Answer: True. Proof. It is closed and bounded in $\mathbb{R}^2$, hence compact.

23. Answer: False. Justification. Although it is bounded, since it is not closed in $\mathbb{R}^2$, it is noncompact.

24. Answer: False. Justification. Although it is closed in $\mathbb{R}^2$, since it is unbounded, it is noncompact.

25. Answer: False. Justification. Although S is closed in $\mathbb{R}$, since it is unbounded, it is noncompact. Alternatively, $\{(-1, n) : n \in \mathbb{N}\}$ is an open cover of S without finite subcover. Hence S is noncompact.

26. Answer: False. Justification. S is not closed in $\mathbb{R}$. In fact, $(\mathbb{R}\backslash\mathbb{Q}) \cap [0,1]$ does not have any interior point, hence $(\mathbb{R}\backslash\mathbb{Q}) \cap [0,1]$ is not open in $[0,1]$ and so $S = [0,1]\backslash((\mathbb{R}\backslash\mathbb{Q}) \cap [0,1])$ is not closed in $[0,1]$, thus noncompact. Alternatively, pick any irrational point $a \in S$. Then

$$\left\{\left(-\frac{1}{n}, a - \frac{1}{n}\right), \left(a - \frac{1}{n}, 1 + \frac{1}{n}\right)\right\}_{n \in \mathbb{N}}$$

is an open cover of S in $\mathbb{R}$ which has no finite subcover. Here, we used the natural convention that $(\alpha, \beta) := \phi$ for $\alpha \geq \beta$.

27. Answer: False. Justification. It is evident that $\left\{\frac{1}{n}\right\}_{n=2}^{\infty}$ is a Cauchy sequence in $(0,1)$ which is not convergent there.

28. Answer: False. Example: Let $X=\mathbb{R}$ with the Euclidean metric. Define

$$x_n := \begin{cases} 0 & \text{if } n \text{ is odd} \\ \frac{1}{n} & \text{if } n \text{ is even.} \end{cases}$$

Then $\{x_n\} \to 0$ and hence in particular, it is Cauchy. However, whenever n, $m \in \mathbb{N}$ are odd and distinct, no matter how large they are, we have

$$d\left(x_{n+1}, x_{m+1}\right) = \left|\frac{1}{n+1} \quad \frac{1}{m+1}\right| > 0 = d\left(x_n, x_m\right)$$

29. It is elementary to verify that d is a well-defined metric on M_{22}. Let $\left\{A_n = \ \ a_{ij}^n)\right\}_{n\in}$ be a Cauchy sequence in (M_{22}, d). By definition, for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\max_{1\leq i,j\leq 2}\left|a_{ij}^n \quad a_{ij}^m\right| = d\left(A_n, A_m\right) < \varepsilon \qquad \text{whenever } n, m \geq N.$$

It follows that for any fixed pair (i,j),

$$\left|a_{ij}^n \quad a_{ij}^m\right| < \epsilon \qquad \text{whenever } n, m \geq N.$$

Hence $\left\{a_{ij}^n\right\}_{n\in}$ is a Cauchy sequence in $\mathbb{R}$, which must be convergent. Let $a_{ij} := \lim_{n\to\infty} a_{ij}^n$. Then $A := (a_{ij})$ is the limit of A_n in d. In fact, for any $\varepsilon > 0$, for any $i, j = 1, 2$, there exists $N_{ij} \in \mathbb{N}$ such that

$$\left|a_{ij}^n \quad a_{ij}\right| < \varepsilon \qquad \text{whenever } n \geq N_{ij}.$$

Take $N := \max_{1\leq i,j\leq 2} N_{ij}$. Then

$$d\left(A_n, A\right) = \max_{1\leq i,j\leq 2}\left|a_{ij}^n \quad a_{ij}\right| < \varepsilon \qquad \text{whenever } n \geq N.$$

30. Answer: True. Proof. Let $\pi_i : \mathbb{R}^2 \to \mathbb{R}$ be the projection of $\mathbb{R}^2$ onto its i-th coordinate, $i = 1, 2$ and π_i is continuous for $i = 1, 2$ (discussed in the class). Define $F : \mathbb{R}^2 \to \mathbb{R}$ by

$$F(x,y) := yf(x) = \left(\pi_2(x,y)\right) \cdot \left(f \circ \pi_1(x,y)\right).$$

Then we have $A = F^{\ 1}((0,\infty))$. Since f, π_1 and π_2 are continuous, so is F. As $(0,\infty) \subset \mathbb{R}$ is open, $A = F^{\ 1}((0,\infty)) \subset \mathbb{R}^2$ is open.

31. Answer: False. Example: Let $X := \mathbb{R}, Y := \{0,1\}$, and $f : X \to Y$ be given by

$$f(x) := \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{if } x \neq 0 \end{cases}$$

Then clearly f sends every compact set to a compact set but f is not continuous.

32. Answer: False. Example: Let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) := \frac{x^2}{x^2+1}, \quad x \in \mathbb{R}.$$

Then $f(\mathbb{R}) = [0,1) \subset \mathbb{R}$ and so in particular, it is neither open nor closed.

33. (a) Example: Define $f(x) =: \begin{cases} 2x & x \in \ \ 0, \frac{1}{2}] \\ 1 & x \in \ \ \frac{1}{2}, 1) \end{cases}.$ It is clear that f is continuous on $(0,1)$ and $f((0,1)) = (0,1]$.

(b) Example: Define $f(x) := \begin{cases} 0 & x \in [0,1] \\ 1 & x \in [2,3]. \end{cases}$ Note that the only open sets in T are $\phi, \{0\}, \{1\}, \{0,1\}$. As

$$f^{\ 1}(\phi) = \phi, \quad f^{\ 1}(\{0\}) = [0,1]$$
$$f^{\ 1}(\{1\}) = [2,3], f^{\ 1}(\{0,1\}) = [0,1] \cup [2,3]$$

are all open in S, we conclude that f is continuous on S.

(c) Since S is compact, T is noncompact, and compactness should be preserved by continuous functions, there cannot be any continuous function mapping S onto T.

34. (a) For any $x \in [0,1] \cap \mathbb{Q}$, take $\varepsilon := \frac{1}{2}$. For any $\delta > 0$, there exists $y \in B_{[0,1]}(x,\delta) \cap (\mathbb{R}\backslash\mathbb{Q})$, and we have $d(f(x), f(y)) = |1 \quad 0| = 1 > \varepsilon$. So f is not continuous at x. Similarly, for any $x \in [0,1] \cap (\mathbb{R}\backslash\mathbb{Q})$, take $\varepsilon := \frac{1}{2}$. For any $\delta > 0$, there exists $y \in B_{[0,1]}(x,\delta) \cap \mathbb{Q}$, and we have $d(f(x), f(y)) = |0 \quad 1| = 1 > \varepsilon$. So f is not continuous at x. Combining, we see that f is not continuous anywhere in $[0,1]$.

(b) For $x = 0$, for any $\varepsilon > 0$, take $\delta := \varepsilon$. Then we have $g \quad B_{[0,1]}(0,\delta)\big) = g([0,\delta)) = g([0,\varepsilon)) \subset [0,\varepsilon) \subset B_{\mathbb{R}}(0,\varepsilon) = B_{\mathbb{R}}(g(0),\varepsilon)$. Hence g is continuous at $x = 0$. For any $x \in (0,1] \cap \mathbb{Q}$, take $\varepsilon := \frac{x}{2}$. For any $\delta > 0$, there exists $y \in B_{[0,1]}(x,\delta) \cap (\mathbb{R}\backslash\mathbb{Q})$, and we have $d(g(x), g(y)) = |x \quad 0| = x > \varepsilon$. So g is not continuous at x. Similarly, for any $x \in (0,1] \cap (\mathbb{R}\backslash\mathbb{Q})$, take $\varepsilon := \frac{x}{2}$. For any $\delta \in \quad 0, \frac{x}{2}\big)$, there exists $y \in B_{[0,1]}(x,\delta) \cap \mathbb{Q}$, and we have $d(g(x), g(y)) = |0 \quad y| = y > x \quad \delta > \frac{x}{2} = \varepsilon$. So g is not continuous at x. Combining, we conclude that g is continuous only at $x = 0$.

(c) h is continuous if and only if $h(x) = c$ for all $x \in [0,1]$. In fact, it is easy to verify that the constant function $h \equiv c$ is continuous. Conversely, let $S = [0,1] \cap \mathbb{Q}$. Then $\bar{S} = [0,1]$ and so for every $x \in [0,1]$, there exists a sequence $\{x_n\}_{n\in}$ in S such that $\{x_n\} \to x$ as $n \to \infty$. By the continuity of h, we have $h(x) = \lim_{n\to\infty} h(x_n) = c$. Hence $h(x) = c$ for all $x \in [0,1]$.

35. (a) It is elementary to verify that D is a well-defined metric on $X \times X$. Let $\{(x_n, y_n)\}_{n\in}$ be a sequence in $X \times X$ such that $\{(x_n, y_n)\} \to (x,y)$ as $n \to \infty$. For any $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that $D((x_n, y_n), (x,y)) < \varepsilon$ for all $n \geq N$. Hence by triangle inequality,

$$\begin{aligned}
&|d(x_n, y_n) \quad d(x,y)| \\
\leq &|d(x_n, y_n) \quad d(x_n, y)| + |d(x_n, y) \quad d(x,y)| \\
\leq &d(y_n, y) + d(x_n, x) \\
\leq &2D((x_n, y_n), (x,y)) \\
< &2\epsilon \text{ for all } n \geq N.
\end{aligned}$$

Therefore, $\{d(x_n, y_n)\} \to d(x,y)$ as $n \to \infty$ and so d is continuous.

(b) Let $\{(x_n, f(x_n))\}_{n\in}$ be a convergent sequence in G, say, $\{(x_n, f(x_n))\} \to (x,y)$ as $n \to \infty$. By the definition of D, we see easily that $\{x_n\} \to x$ and $\{f(x_n)\} \to y$ as $n \to \infty$. As f is continuous, we have $\{f(x_n)\} \to f(x)$ as $n \to \infty$ and so $f(x) = y$. Hence $\{(x_n, f(x_n))\} \to (x, f(x)) \in G$ as $n \to \infty$. Therefore, G is closed in $X \times X$.

36. (a) (i) True.

Proof. For any open set $U \subset \mathbb{R}^2$ and any $x \in \pi(U) \subset \mathbb{R}$, there exists $y \in \mathbb{R}$ such that $(x,y) \in U$. Since $U \subset \mathbb{R}^2$ is open, there exists $r > 0$ such that $B_{\mathbb{R}^2}((x,y),r) \subset U$. Hence $x \in (x \quad r, x+r) = B_{\mathbb{R}}(x,r) \subset \pi(B_{\mathbb{R}^2}((x,y),r)) \subset \pi(U)$. Therefore, $\pi(U)$ is open and so π is open.

(ii) False.

Example: $S := \{ \quad x, \frac{1}{x}) : x > 0\} \subset \mathbb{R}^2$ is closed but $\pi(S) = (0,\infty) \subset \mathbb{R}$ is not closed.

(b) Answer: False. Example: Let $X := (0,1) \subset \mathbb{R}$ and $Y := \mathbb{R}$ with the usual Euclidean metric. Consider $f(x) := x, x \in X$. As open sets in $(0,1)$ are open in $\mathbb{R}, f$ is open. It is trivial that f is continuous. However, as the image of the closed set $(0,1) \subset X$ equals $(0,1) \subset Y$ which is not closed in Y, f is not closed.

(c) Answer: False. Example: Let $X := \{0\} \subset \mathbb{R}$ and $Y := \mathbb{R}$ with the usual Euclidean metric. Consider $f(x) := x, x \in X$. Then it is evident that f is closed and continuous. However, as the image of the open set $\{0\} \subset X$ equals $\{0\} \subset Y$ which is not open in Y, f is not open.

(d) Answer: False. Example: Let $X := \mathbb{R}$ with the usual Euclidean metric and let $Y := \mathbb{R}$ with the discrete metric. Consider $f(x) := x, x \in X$. As all subsets of Y are clopen, f is both open and closed. However, as the pre-image of the open set $\{0\} \subset Y$ equals $\{0\} \subset X$ which is not open in X, f is not continuous.

37. (a) Answer: True.

Proof. Consider the function $f : (0,1) \to \mathbb{R}$ given by

$$f(x) := \tan\left(\pi\left(x \quad \frac{1}{2}\right)\right), \quad x \in (0,1).$$

It is evident that f is $1-1$, onto, continuous, with continuous inverse $f^{-1}:\mathbb{R}\to(0,1)$ given by

$$f^{-1}(y) := \frac{1}{\pi}\tan^{-1} y + \frac{1}{2}, \quad y \in \mathbb{R}.$$

Thus $(0,1)\cong\mathbb{R}$.

(b) Answer: False.

Justification. Being a closed and bounded subset of $\mathbb{R}^2, S^1$ is compact. Since compactness is a topological property, S^1 cannot be homeomorphic to the non-compact space $\mathbb{R}$.

(c) Answer: False.

Justification. Write $X = (0,1)\cup\{2\}$. Suppose there were a homeomorphism $f : X \to (0,1]$. Then as $(0,1)\subset X$ is closed, $f((0,1))\subset(0,1]$ should also be closed. However, observe that

$$\begin{aligned} f((0,1)) &= (0,1]\backslash\{f(2)\} \\ &= \begin{cases} (0,1) & \text{if } f(2)=1 \\ (0,a)\cup(a,1] & \text{if } f(2)=a\in(0,1). \end{cases} \end{aligned}$$

In either case, $f((0,1))$ is not closed in $(0,1]$.

38. (a) As $[0,1]$ and $[2,3]$ are nonempty disjoint open subsets of X whose union is X, they form a separation of X.

(b) As $[0,1)$ and $(1,2]$ are nonempty disjoint open subsets of X whose union is X, they form a separation of X.

(c) As $(-\infty,0)$ and $(0,\infty)$ are nonempty disjoint open subsets of X whose union is X, they form a separation of X.

(d) Take any $x\in\mathbb{R}\backslash\mathbb{Q}$. Then $\{(-\infty,x)\cap\mathbb{Q},(x,\infty)\cap\mathbb{Q}\}$ is a separation of $\mathbb{Q}$.

(e) Let X be a discrete metric space with $\#(X)>1$ and $a\in X$. Then $X\backslash\{a\}\neq\phi$ and so $\{\{a\},X\backslash\{a\}\}$ is a separation of X.

(f) Since a singleton cannot be split into two nonempty subsets, it is clear that every singleton is connected. In particular, every metric space contains connected subsets.

(g) Without loss of generality, we consider $S^1\backslash\{n,s\}$, where $n:=(0,1)$ denotes the "north pole" and $s:=(0,-1)$ the "south pole" of the unit circle S^1. Then

$$\left\{S^1\cap\left\{(x,y)\in\mathbb{R}^2 : x>0\right\}, S^1\cap\left\{(x,y)\in\mathbb{R}^2 : x<0\right\}\right\}$$

is a separation of $S^1\backslash\{n,s\}$.

(h) Without loss of generality, we take S^1 as the unit circle $\left\{(x,y,0) : x^2+y^2=1\right\}$ in the xy-plane in $\mathbb{R}^3$. Then

$$\left\{S^2\cap\left\{(x,y,z)\in\mathbb{R}^3 : z>0\right\}, \left\{S^2\cap\left\{(x,y,z)\in\mathbb{R}^3 : z<0\right\}\right\}\right.$$

is a separation of $S^2\backslash S^1$.

39. (a) Answer: False.

Example: Let $X:=\mathbb{R}, S:=[0,1], T:=(1,2)$. Clearly S and T are disjoint but $S\cup T=[0,2)$ is connected.

(b) Answer: False.

Example: Let $X:=\mathbb{R}$ and $S:=\{0\}$. Then S is connected but $S\not\subset\phi=S'$.

(c) Answer: False.

Example: Let $X:=\mathbb{R}$ and $S:=(0,1)$. Then S is connected but $\partial S=\{0,1\}$ is not.

(d) Answer: False. Example: Let $X:=\mathbb{R}$ and $S:=(0,1)\cup\{2\}$. Then $S\backslash S'=\{2\}$ is connected but S is not.

(e) Answer: False. Example: Let $X:=\mathbb{R}\backslash\{0\}$. Then $B(1,2)=(-1,0)\cup(0,3)$ is disconnected.

40. (a) For any $\varepsilon > 0$ and any $x_1, x_2 \in (0, 1]$, we have

$$|f(x_1) - f(x_2)| = \left|x_1^2 - x_2^2\right| = |x_1 + x_2|\,|x_1 - x_2| \le 2\,|x_1 - x_2|\,.$$

Hence whenever $2\,|x_1 - x_2| < \varepsilon$, we have $|f(x_1) - f(x_2)| < \varepsilon$. This suggests that $\delta := \varepsilon/2$ is a uniform δ that works everywhere on $(0, 1]$.

So the formal proof goes as follows: For any $\varepsilon > 0$, pick $\delta := \varepsilon/2$. Then we have

$$|f(x_1) - f(x_2)| = \left|x_1^2 - x_2^2\right| = |x_1 + x_2|\,|x_1 - x_2| \le 2\,|x_1 - x_2| < 2\delta = \varepsilon$$

for all $x_1, x_2 \in S$ with $|x_1 - x_2| < \delta$. Hence f is uniformly continuous on $(0, 1]$.

(b) It is clear that f is continuous everywhere on $\mathbb{R}$. Also, by the Remark following Example 4.1.4, f is uniformly continuous on every bounded subset of $\mathbb{R}$. However, it is not uniformly continuous on $(0, \infty)$. In fact, in view of the preceding Remark, let us take $\varepsilon := 1$. For any $\delta > 0$, take $x_1 := \frac{1}{\delta}$ and $x_2 := \frac{1}{\delta} + \frac{\delta}{2}$. Then $x_1, x_2 \in (0, \infty)$