Hints / Answers / Solutions

1. Let c denote the probability of a single odd face. Then the probability of a single even face is $2c$, and by adding the probabilities of the 3 odd faces and the 3 even faces, we get $9c = 1$. Thus, $c = 1/9$. The desired probability is
$$\mathbf{P}(\{1,2,3\}) = \mathbf{P}(\{1\}) + \mathbf{P}(\{2\}) + \mathbf{P}(\{3\}) = c + 2c + c = 4c = 4/9.$$

2. The easiest way to solve this problem is to make a table of some sort, similar to the one below.

Die 1	Die 2	Sum	**P**(Sum)
1	1	2	2 p
1	2	3	3 p
1	3	4	4 p
1	4	5	5 p
2	1	3	3 p
2	2	4	4 p
2	3	5	5 p
2	4	6	6 p
3	1	4	4 p
3	2	5	5 p
3	3	6	6 p
3	4	7	7 p
4	1	5	5 p
4	2	6	6 p
4	3	7	7 p
4	4	8	8 p
		Total	80 p

$\mathbf{P}$(All outcomes) $= 80p$ (Total from the table)

and therefore $p = \frac{1}{80}$

(a)

$$\mathbf{P}(\text{ Even sum }) = 2p + 4p + 4p + 6p + 4p + 6p + 6p + 8p = 40p = 1/2$$

(b)

$$\mathbf{P}(\text{ Rolling a 2 and a 3}) = \mathbf{P}(2,3) + \mathbf{P}(3,2) = 5p + 5p = 10p = 1/8$$

3. (a) The probability of Mike scoring 50 points is proportional to the area of the inner disk. Hence, it is equal to $\alpha\pi R^2 = \alpha\pi$, where α is a constant to be determined. Since the probability of landing the dart on the board is equal to one, $\alpha\pi 10^2 = 1$, which implies that $\alpha = 1/(100\pi)$. Therefore, the probability that Mike scores 50 points is equal to $\pi/(100\pi) = 0.01$

   (b) In order to score exactly 30 points, Mike needs to place the dart between 1 and 3 inches from the origin. An easy way to compute this probability is to look first at that of scoring more than 30 points, which is equal to $\alpha\pi 3^2 = 0.09$. Next, since the 30 points ring is disjoint from the 50 points disc, probability of scoring more than 30 points is equal to the probability of scoring 50 points plus that of scoring exactly 30 points. Hence, the probability of Mike scoring exactly 30 points is equal to $0.09 - 0.01 = 0.08$

   (c) For the part (a) question. The probability of John scoring 50 points is equal to the probability of throwing in the right half of the board and scoring 50 points plus that of throwing in the left half and scoring 50 points. The first term in the sum is proportional to the area of the right half of the inner disk and is equal to $\alpha\pi R^2/2 = \alpha\pi/2$, where α is a constant to be determined. Similarly, the probability of him throwing in the left half of the board and scoring 50 points is equal to $\beta\pi/2$, where β is a constant (not necessarily equal to α). In order to determine α and β, let us compute the probability of throwing the dart in the right half of the board. This probability is equal to
$$\alpha\pi R^2/2 = \alpha\pi 10^2/2 = \alpha 50\pi$$
   Since that probability is equal to $2/3, \alpha = 1/(75\pi)$. In a similar fashion, β can be determined to be $1/(150\pi)$. Consequently, the total probability is equal to $1/150 +\ 1/300 = 0.01$

For the part (b), The probability of scoring exactly 30 points is equal to that of scoring more than 30 points minus that of scoring exactly 50. By applying the same type of analysis as in (b) above, the probability is found to be equal to 0.08.

These numbers suggest that John and Mike have similar skills, and are equally likely to win the game. The fact that Mike's better control (or worst, depending on how you look at it) of the direction of his throw does not increase his chances of winning can be explained by the observation that both players' control over the distance from the origin is identical.

4. Let A be the event that the first toss is a head and let B be the event that the second toss is a head. We must compare the conditional probabilities $\mathbf{P}(A \cap B \mid A)$ and $\mathbf{P}(A \cap B \mid A \cup B)$. We have

$$\mathbf{P}(A \cap B \mid A) = \frac{\mathbf{P}((A \cap B) \cap A)}{\mathbf{P}(A)} = \frac{\mathbf{P}(A \cap B)}{\mathbf{P}(A)}$$

and

$$\mathbf{P}(A \cap B \mid A \cup B) = \frac{\mathbf{P}((A \cap B) \cap (A \cup B))}{\mathbf{P}(A \cup B)} = \frac{A \cap B}{A \cup B}.$$

Since $\mathbf{P}(A \cup B) \geq \mathbf{P}(A)$, the first conditional probability above is at least as large, so Alice is right, regardless of whether the coin is fair or not. In the case where the coin is fair, that is, if all four outcomes HH, HT, TH, TT are equally likely, we have

$$\frac{\mathbf{P}(A \cap B)}{\mathbf{P}(A)} = \frac{1/4}{1/2} = \frac{1}{2}, \quad \frac{\mathbf{P}(A \cap B)}{\mathbf{P}(A \cup B)} = \frac{1/4}{3/4} = 1/3.$$

A generalization of Alice's reasoning is that if A, B, and C are events such that $B \subset C$ and $A \cap B = A \cap C$ (for example, if $A \subset B \subset C$), then the event A is at least as likely if we know that B has occurred than if we know that C has occurred. Alice's reasoning corresponds to the special case where $C = A \cup B$.

5. (a) Let A_i be the event of playing with an opponent of type i. We have

$$\mathbf{P}(A_1) = 0.5, \quad \mathbf{P}(A_2) = 0.25. \quad \mathbf{P}(A_3) = 0.25$$

Also. let B be the event of winning. We have

$$\mathbf{P}(B \mid A_1) = 0.3. \quad \mathbf{P}(B \mid A_2) = 0.4. \quad \mathbf{P}(B \mid A_3) = 0.5$$

Thus, by the total probability theorem, the probability of winning is

$$\begin{aligned} \mathbf{P}(B) &= \mathbf{P}(A_1)\mathbf{P}(B \mid A_1) + \mathbf{P}(A_2)\mathbf{P}(B \mid A_2) + \mathbf{P}(A_3)\mathbf{P}(B \mid A_3) \\ &= 0.5 \cdot 0.3 + 0.25 \cdot 0.4 + 0.25 \cdot 0.5 \\ &= 0.375. \end{aligned}$$

(b) Here, A_2 is the event of getting an opponent of type i, and

$$\mathbf{P}(A_1) = 0.5, \quad \mathbf{P}(A_2) = 0.25. \quad \mathbf{P}(A_3) = 0.25$$

Also, B is the event of winning, and

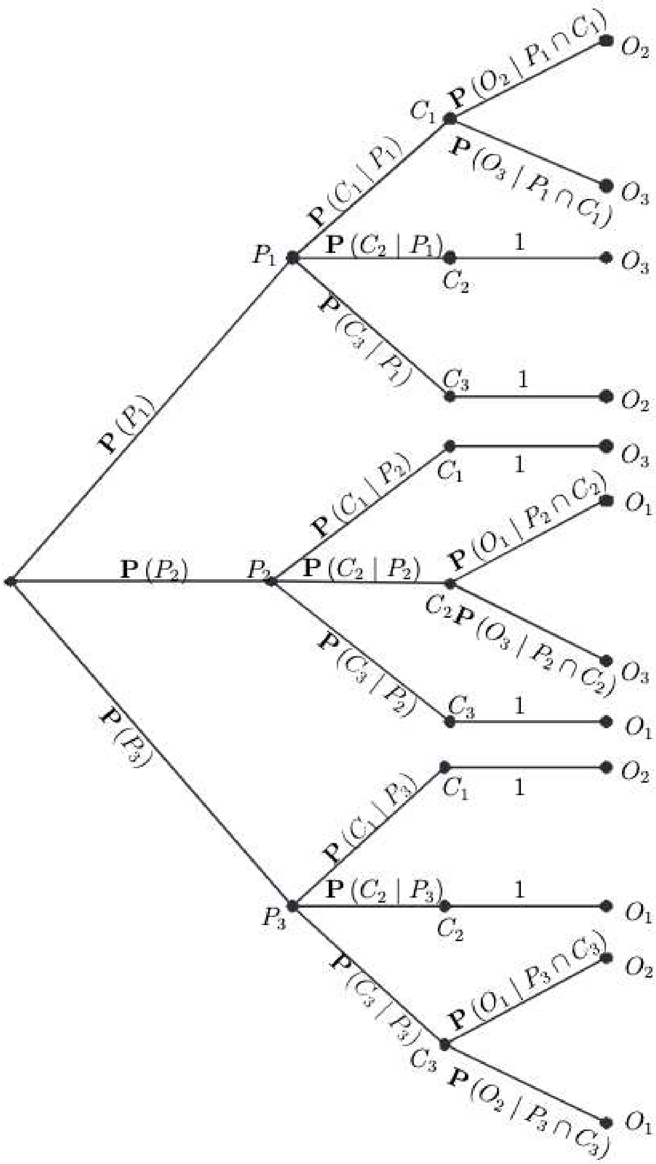
$$\mathbf{P}(B \mid A_1) = 0.3. \quad \mathbf{P}(B \mid A_2) = 0.4, \quad \mathbf{P}(B \mid A_3) = 0.5$$

Suppose that you win. What is the probability $\mathbf{P}(A_1 \mid B)$ that you had an opponent of type 1 ?

Using Bayes' rule, we have

$$\begin{aligned} \mathbf{P}(A_1 \mid B) &= \frac{\mathbf{P}(A_1)\mathbf{P}(B \mid A_1)}{\mathbf{P}(A_1)\mathbf{P}(B \mid A_1) + \mathbf{P}(A_2)\mathbf{P}(B \mid A_2) + \mathbf{P}(A_3)\mathbf{P}(B \mid A_3)} \\ &= \frac{0.5 \cdot 0.3}{0.5 \cdot 0.3 + 0.25 \cdot 0.4 + 0.25 \cdot 0.5} \\ &= 0.4. \end{aligned}$$

6. Let P_i denote the event where the prize is behind door i, C_i denote the event where you initially choose door i, and O_i denote the event where your friend opens door i. The corresponding probability tree



is:

(a) The probability of winning when not switching from your initial choice is the probability that the prize is behind the door you initially chose:

$$
\begin{aligned}
\mathbf{P}(\text{ Win when not switching }) &= \mathbf{P}(P_1 \cap C_1) + \mathbf{P}(P_2 \cap C_2) + \mathbf{P}(P_3 \cap C_3) \\
&= \mathbf{P}(P_1)\,\mathbf{P}(C_1 \mid P_1) + \mathbf{P}(P_2)\,\mathbf{P}(C_2 \mid P_2) + \mathbf{P}(P_3)\,\mathbf{P}(C_3 \mid P_3) \\
&= \mathbf{P}(P_1)\,\mathbf{P}(C_1) + \mathbf{P}(P_2)\,\mathbf{P}(C_2) + \mathbf{P}(P_3)\,\mathbf{P}(C_3) \\
&= 1/3 \cdot (\mathbf{P}(C_1) + \mathbf{P}(C_2) + \mathbf{P}(C_3)) \\
&= 1/3
\end{aligned}
$$

(b) The probability of winning when switching from your initial choice is the probability that the prize is behind the remaining (unopened) door:

$$
\begin{aligned}
\mathbf{P}(\text{ Win when switching }) = & \mathbf{P}(P_1 \cap C_2 \cap O_3)+\mathbf{P}(P_1 \cap C_3 \cap O_2)+\mathbf{P}(P_2 \cap C_1 \cap O_3) \\
& +\mathbf{P}(P_2 \cap C_3 \cap O_1)+\mathbf{P}(P_3 \cap C_1 \cap O_2)+\mathbf{P}(P_3 \cap C_2 \cap O_1) \\
= & \mathbf{P}(P_1 \cap C_2)+\mathbf{P}(P_1 \cap C_3)+\mathbf{P}(P_2 \cap C_1)+\mathbf{P}(P_2 \cap C_3) \\
& +\mathbf{P}(P_3 \cap C_1)+\mathbf{P}(P_3 \cap C_2) \\
= & \mathbf{P}(P_1)\mathbf{P}(C_2)+\mathbf{P}(P_1)\mathbf{P}(C_3)+\mathbf{P}(P_2)\mathbf{P}(C_1)+\mathbf{P}(P_2)\mathbf{P}(C_3) \\
& +\mathbf{P}(P_3)\mathbf{P}(C_1)+\mathbf{P}(P_3)\mathbf{P}(C_2) \\
= & 2/3 \cdot (\mathbf{P}(C_1)+\mathbf{P}(C_2)+\mathbf{P}(C_3)) \\
= & 2/3
\end{aligned}
$$

(c) Given C_1, that you first choose door 1 , with the new strategy of switching only if door 3 is opened, you win if the prize behind door 1 and door 2 is opened or if the prize is behind door 2 and door 3 is opened.

$$
\begin{aligned}
\mathbf{P}(\text{ Win with new strategy } \mid C_1) & = \mathbf{P}(P_1 \cap O_2 \mid C_1)+\mathbf{P}(P_2 \cap O_3 \mid C_1) \\
& = \mathbf{P}(P_1 \mid C_1)\mathbf{P}(O_2 \mid P_1 \cap C_1)+\mathbf{P}(P_2 \mid C_1)\mathbf{P}(O_3 \mid P_2 \cap C_1) \\
& = \mathbf{P}(P_1)\mathbf{P}(O_2 \mid P_1 \cap C_1)+\mathbf{P}(P_2)\mathbf{P}(O_3 \mid P_2 \cap C_1) \\
& = 1/3 \cdot \mathbf{P}(O_2 \mid P_1 \cap C_1)+1/3 \cdot 1 \\
& = 1/3 \cdot (\mathbf{P}(O_2 \mid P_1 \cap C_1)+1)
\end{aligned}
$$

Given that your initial choice is door 1 , the probability of winning under this new strategy is dependent on how your friend decides which of doors 2 or 3 to open if the prize also lies behind door 1. If he always picks door 2, then $\mathbf{P}(O_2 \mid P_1 \cap C_1)=1$ and $\mathbf{P}(\text{ Win with new strategy } \mid C_1)=2/3$. If he picks between doors 2 and 3 with equal probability then $\mathbf{P}(O_2 \mid P_1 \cap C_1)=1/2$ and $\mathbf{P}(\text{ Win with new strategy } \mid C_1)=1/2$.

Explaining in words: Under the strategy of no switching, your initial choice will determine whether you win or not, and the probability of winning is 1/3. This is because the prize is equally likely to be behind each door.

Under the strategy of switching, if the prize is behind the initially chosen door (probability 1/3). you do not win. If it is not (probability 2/3), and given that another door without a prize has been opened for you, you will get to the winning door once you switch. Thus. the probability of winning is now 2/3, so (b) is a better strategy than (a).

Consider now strategy (c). Under this strategy, there is insufficient information for determining the probability of winning. The answer depends on the way that your friend chooses which door to open. Let us consider two possibilities.

Suppose that if the prize is behind door 1, your friend always chooses to open door 2. (If the prize is behind door 2 or 3 , your friend has no choice.) If the prize is behind door 1 . your friend opens door 2 , you do not switch, and you win. If the prize is behind door 2, your friend opens door 3, you switch. and you win. If the prize is behind door 3. your friend opens door 2 . you do not switch, and you lose. Thus, the probability of winning is 2/3. so strategy (c) in this case is as good as strategy (b).

Suppose now that if the prize is behind door 1. your friend is equally likely to open either door 2 or 3 . If the prize is behind door 1 (probability 1/3). and if your friend opens door 2 (probability 1/2), you do not switch and you win (probability 1/6). But if your friend opens door 3 , you switch and you lose. If the prize is behind door 2 , your friend opens door 3 . you switch, and you win (probability 1/3). If the prize is behind door 3, your friend opens door 2, you do not switch and you lose. Thus. the probability of winning is $1/6+1/3=1/2$, so strategy (c) in this case is inferior to strategy (b).

7. (a) The events H_1 and H_2 are (unconditionally) independent.

   (b) But

$$
\mathbf{P}(H_1 \mid D)=\frac{1}{2}, \quad \mathbf{P}(H_2 \mid D)=\frac{1}{2}, \quad \mathbf{P}(H_1 \cap H_2 \mid D)=0
$$

   so that $\mathbf{P}(H_1 \cap H_2 \mid D) \neq \mathbf{P}(H_1 \mid D)\mathbf{P}(H_2 \mid D)$, and H_1, H_2 are not conditionally independent.

8. (a) In order to wind up in the same place after two steps, the tightrope walker can either step forwards, then backwards, or vice versa. Therefore the required probability is:

$$
2 \cdot p \cdot (1-p)
$$

(b) The probability that after three steps he will be one step ahead of his starting point is the probability that out of 3 steps in total, 2 of them are forwards, and one is backwards. This equals:

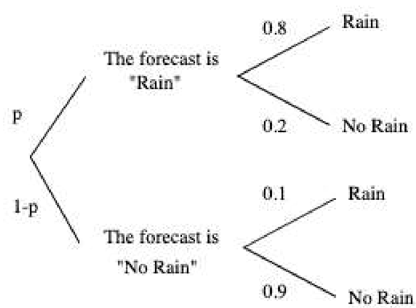
$$3 \cdot p^2 \cdot (1-p)$$

(c) Given that out of his three steps only one is backwards, the sample space for the experiment is:

$$\{(F, F, B); (F, B, F); (B, F, F)\}$$

where F denotes a step forwards, and B a step backwards. Each of these sample points is equally likely, therefore the probability that his first step is a step forward is $\frac{2}{3}$.

9. (a) The tree representation during the winter can be drawn as the following:



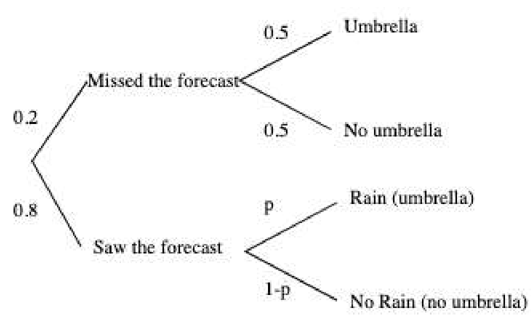
Let A be the event that the forecast was **Rain,** let B be the event that it rained, and let p be the probability that the forecast says **Rain**. If it is in the winter, $p = 0.7$ and

$$\mathbf{P}(A \mid B) = \frac{\mathbf{P}(B \mid A)\mathbf{P}(A)}{\mathbf{P}(B)} = \frac{(0.8)(0.7)}{(0.8)(0.7) + (0.1)(0.3)} = \frac{56}{59}$$

Similarly, if it is in the summer, $p = 0.2$ and

$$\mathbf{P}(A \mid B) = \frac{\mathbf{P}(B \mid A)\mathbf{P}(A)}{\mathbf{P}(B)} = \frac{(0.8)(0.2)}{(0.8)(0.2) + (0.1)(0.8)} = \frac{2}{3}$$

(b) Let C be the event that Victor is carrying an umbrella. Let D be the event that the forecast is no rain. The tree diagram in this case is:



$\mathbf{P}(D) = 1 - p$

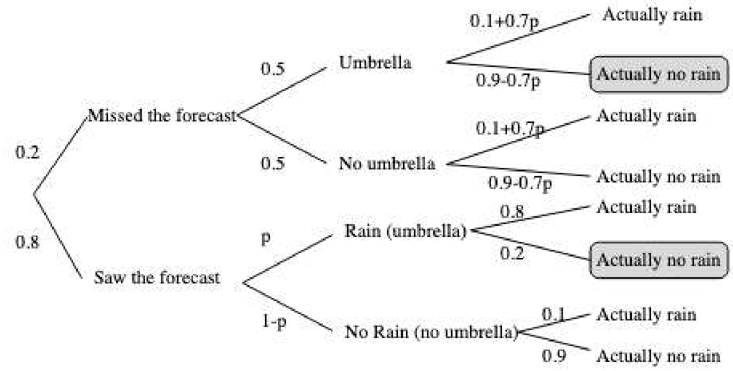
$\mathbf{P}(C) = (0.8)p + (0.2)(0.5) = 0.8p + 0.1$

$\mathbf{P}(C \mid D) = (0.8)(0) + (0.2)(0.5) = 0.1$

(c) Let us first find the probability of rain if Victor missed the forecast.

$$\mathbf{P} \text{ (actually rains } \mid \text{ missed forecast)} = (0.8)p + (0.1)(1-p) = 0.1 + 0.7p$$

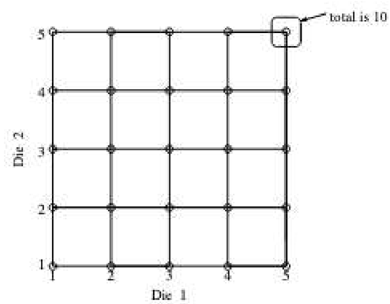
Then, we can extend the tree in part (b) as follows:



Therefore, given that Victor is carrying an umbrella and it is not raining, we are looking at the two shaded cases.

$$\mathbf{P}(\text{ saw forecast } \mid \text{ umbrella and not raining }) = \frac{(0.8)p(0.2)}{(0.8)p(0.2) + (0.2)(0.5)(0.9 - 0.7p)}$$

In fall and winter, $p = 0.7$, so the probability is $\frac{112}{153}$. In summer and spring, $p = 0.2$, so the probability is $\frac{8}{27}$.



10. (a) (i) No. Overall, there are 25 different outcomes in the sample space. For a total of 10 , we should get a 5 on both rolls. Therefore $A \subset B$, and

$$\mathbf{P}(B \mid A) = \frac{\mathbf{P}(A \cap B)}{\mathbf{P}(A)} = \frac{\mathbf{P}(A)}{\mathbf{P}(A)} = 1$$

We observe that to get at least one 5 showing, we can have 5 on the first roll, 5 on the second roll, or 5 on both rolls, which corresponds to 9 distinct outcomes in the sample space. Therefore

$$\mathbf{P}(B) = \frac{9}{25} \neq \mathbf{P}(B \mid A)$$

ii. No. Given event A, we know that both roll outcomes must be 5 . Therefore, we could not have event C occur, which would require at least one 1 showing. Formally, there are 9 outcomes in C, and

$$\mathbf{P}(C) = \frac{9}{25}$$

But

$$\mathbf{P}(C \mid A) = 0 \neq \mathbf{P}(C)$$

(b) i. No. Out of the total 25 outcomes, 5 outcomes correspond to equal numbers in the two rolls. In half of the remaining 20 outcomes, the second number is higher than the first one. In the other half, the first number is higher than the second. Therefore,

$$\mathbf{P}(F) = \frac{10}{25}$$

There are eight outcomes that belong to event E :

$$E = \{(1,2),(2,3),(3,4),(4,5),(2,1),(3,2),(4,3),(5,4)\}$$

To find $\mathbf{P}(F \mid E)$, we need to compute the proportion of outcomes in E for which the second number is higher than the first one:

$$\mathbf{P}(F \mid E) = \frac{1}{2} \neq \mathbf{P}(F)$$

ii. Yes. Conditioning on event D reduces the sample space to just four outcomes

$$\{(2,5),(3,4),(4,3),(5,2)\}$$

which are all equally likely. It is easy to see that

$$\mathbf{P}(E \mid D)=\frac{2}{4}=\frac{1}{2}, \quad \mathbf{P}(F \mid D)=\frac{2}{4}=\frac{1}{2}, \quad \mathbf{P}(E \cap F \mid D)=\frac{1}{4}=\mathbf{P}(E \mid D)\mathbf{P}(F \mid D)$$

11. (a)

$$\begin{aligned}
\mathbf{P}(\text{ find in A and in A}) &= \mathbf{P}(\text{ in A}) \cdot \mathbf{P}(\text{ find in A} \mid \text{ in A}) = 0.4 \cdot 0.25 = 0.1 \\
\mathbf{P}(\text{ find in B and in B}) &= \mathbf{P}(\text{ in B}) \cdot \mathbf{P}(\text{ find in B} \mid \text{ in B}) = 0.6 \cdot 0.15 = 0.09
\end{aligned}$$

Oscar should search in Forest A first.

(b) Using Bayes' Rule,

$$\begin{aligned}
\mathbf{P}(\text{ in A} \mid \text{ not find in A}) &= \frac{\mathbf{P}(\text{ not find in A} \mid \text{ in A}) \cdot \mathbf{P}(\text{ in A})}{\mathbf{P}(\text{ not find in A} \mid \text{ in A}) \cdot \mathbf{P}(\text{ in A}) + \mathbf{P}(\text{ not find in A} \mid \text{ in B}) \cdot \mathbf{P}(\text{ in B})} \\
&= \frac{(0.75)\cdot(0.4)}{(0.4)\cdot(0.75)+(1)\cdot(0.6)} = \frac{1}{3}
\end{aligned}$$

(c) Again, using Bayes' Rule,

$$\begin{aligned}
\mathbf{P}(\text{ looked in A} \mid \text{ find dog }) &= \frac{\mathbf{P}(\text{ find dog} \mid \text{ looked in A}) \cdot \mathbf{P}(\text{ looked in A})}{\mathbf{P}(\text{ find dog})} \\
&= \frac{(0.25)\cdot(0.4)\cdot(0.5)}{(0.25)\cdot(0.4)\cdot(0.5)+(0.15)\cdot(0.6)\cdot(0.5)} = \frac{10}{19}
\end{aligned}$$

(d) In order for Oscar to find the dog, it must be in Forest A, not found on the first day, alive, and found on the second day. Note that this calculation requires conditional independence of not finding the dog on different days and the dog staying alive.

$$\begin{aligned}
\mathbf{P}(\text{ find live dog in A day 2}) =& \mathbf{P}(\text{ in A}) \cdot \mathbf{P}(\text{ not find in A day 1} \mid \text{ in A}) \\
& \cdot \mathbf{P}(\text{ alive day 2}) \cdot \mathbf{P}(\text{ find day 2} \mid \text{ in A}) \\
=& 0.4 \cdot 0.75 \cdot \left(1-\frac{1}{3}\right) \cdot 0.25 = 0.05
\end{aligned}$$

12. (a) Suppose we choose old widgets. Before we choose any widgets, there are $500 \cdot 0.15 = 75$ defective old widgets. The probability that we choose two defective widgets is

$$\begin{aligned}
\mathbf{P}(\text{ two defective} \mid \text{old }) &= \mathbf{P}(\text{ first is defective} \mid \text{old }) \cdot \mathbf{P}(\text{ second is defective} \mid \text{first is defective, old }) \\
&= \frac{75}{500}\frac{74}{499} = 0.02224
\end{aligned}$$

Now let's consider the new widgets. Before we choose any widgets, there are $1500 \cdot 0.05 = 75$ defective old widgets. Similar to the calculations above,

$$\begin{aligned}
\mathbf{P}\ (\text{two defective} \mid \text{new}) &= \mathbf{P}\ (\text{first is defective} \mid \text{new}) \cdot \mathbf{P}\ (\text{second is defective} \mid \text{first is defective, new}) \\
&= \frac{75}{1500}\frac{74}{1499} = 0.002568
\end{aligned}$$

By the total probability law,

$$\begin{aligned}
\mathbf{P}(\text{ two defective }) =& \mathbf{P}(\text{ old }) \cdot \mathbf{P}(\text{ two defective} \mid \text{ old }) \\
& + \mathbf{P}(\text{ new }) \cdot \mathbf{P}(\text{ two defective} \mid \text{ new }) \\
=& \frac{1}{2} \cdot 0.02224 + \frac{1}{2} \cdot 0.002568 = 0.01240
\end{aligned}$$

Note that this number is very close to what we would get if we ignored the effects of removing one defective widget before choosing the second widget:

$$\begin{aligned}\mathbf{P}(\text{ two defective }) &= \mathbf{P}(\text{ old }) \cdot \mathbf{P}(\text{ two defective } \mid \text{ old }) \\ &\quad + \mathbf{P}(\text{ new }) \cdot \mathbf{P}(\text{ two defective } \mid \text{ new }) \\ &\approx \frac{1}{2} \cdot 0.15^2 + \frac{1}{2} \cdot 0.05^2 = 0.0125\end{aligned}$$

(b) Using Bayes' rule,

$$\begin{aligned}\mathbf{P}(\text{ old } \mid \text{ two defective }) &= \frac{\mathbf{P}(\text{ old }) \cdot \mathbf{P}(\text{ two defective } \mid \text{ old })}{\mathbf{P}(\text{ old }) \cdot \mathbf{P}(\text{ two defective } \mid \text{ old }) + \mathbf{P}(\text{ new }) \cdot \mathbf{P}(\text{ two defective } \mid \text{ new })} \\ &= \frac{\frac{1}{2} \cdot 0.02224}{\frac{1}{2} \cdot 0.02224 + \frac{1}{2} \cdot 0.002568} = 0.8965\end{aligned}$$

13. (a) X is a Binomial random variable with $n = 10, p = 0.2$. Therefore,

$$p_X(k) = \binom{10}{k} 0.2^k 0.8^{10-k}, \qquad \text{for } k = 0, \ldots, 10$$

and $p_X(k) = 0$ otherwise.

(b) $\mathbf{P}(\text{ No hits }) = p_X(0) = (0.8)^{10} = 0.1074$

(c) $\mathbf{P}$ (More hists than misses) $= \sum_{k=6}^{10} p_X(k) = \sum_{k=6}^{10} \binom{10}{k} 0.2^k 0.8^{10-k} = 0.0064$

(d) Since X is a Binomial random variable,

$$\mathbf{E}[X] = 10 \cdot 0.2 = 2 \quad \text{var}(X) = 10 \cdot 0.2 \cdot 0.8 = 1.6$$

(e) $Y = 2X - 3$, and therefore

$$\mathbf{E}[Y] = 2\mathbf{E}[X] - 3 = 1 \quad \text{var}(Y) = 4\,\text{var}(X) = 6.4$$

(f) $Z = X^2$, and therefore

$$\mathbf{E}[Z] = \mathbf{E}\left[X^2\right] = (\mathbf{E}[X])^2 + \text{var}(X) = 5.6$$

14. (a) We expect $\mathbf{E}[X]$ to be higher than $\mathbf{E}[Y]$ since if we choose the student, we are more likely to pick a bus with more students.

(b) To solve this problem formally, we first compute the PMF of each random variable and then compute their expectations.

$$p_X(x) = \begin{cases} 40/148 & x = 40 \\ 33/148 & x = 33 \\ 25/148 & x = 25 \\ 50/148 & x = 50 \\ 0 & \text{otherwise.} \end{cases}$$

and $\mathbf{E}[X] = 40\frac{40}{148} + 33\frac{33}{148} + 25\frac{25}{148} + 50\frac{50}{148} = 39.28$

$$p_Y(y) = \begin{cases} 1/4 & y = 40, 33, 25, 50 \\ 0 & \text{otherwise} \end{cases}$$

and $\mathbf{E}[Y] = 40\frac{1}{4} + 33\frac{1}{4} + 25\frac{1}{4} + 50\frac{1}{4} = 37$ Clearly, $\mathbf{E}[X] > \mathbf{E}[Y]$.

15. The hats of n persons are thrown into a box. The persons then pick up their hats at random (i.e., so that every assignment of the hats to the persons is equally likely). What is the probability that

(a) every person gets his or her hat back?

Answer: $\frac{1}{n!}$.

Solution: consider the sample space of all possible hat assignments. It has n ! elements (n hat selections for the first person, after that $n-1$ for the second, etc.), with every singleelement event equally likely (hence having probability $1/n!$). The question is to calculate the probability of a single-element event, so the answer is $1/n$!

(b) the first m persons who picked hats get their own hats back?

Answer: $\frac{(n-m)!}{n!}$.

Solution: consider the same sample space and probability as in the solution of (a). The probability of an event with $(n-m)$! elements (this is how many ways there are to disribute the remaining $n-m$ hats after the first m are assigned to their owners) is $(n-m)!/n$!

(c) everyone among the first m persons to pick up the hats gets back a hat belonging to one of the last m persons to pick up the hats?

Answer: $\frac{m!(n-m)!}{n!} = \frac{1}{\binom{n}{m}} = \frac{1}{\binom{n}{n-m}}$.

Solution: there are m ! ways to distribute m hats among the first m persons, and ($n-m$)! ways to distribute the remaining $n-m$ hats. The probability of an event with $m!(n-m)$! elements is $m!(n-m)!/n!$.

Now assume, in addition, that every hat thrown into the box has probability p of getting dirty (independently of what happens to the other hats or who has dropped or picked it up). What is the probability that

(d) the first m persons will pick up clean hats?

Answer: $(1-p)^m$.

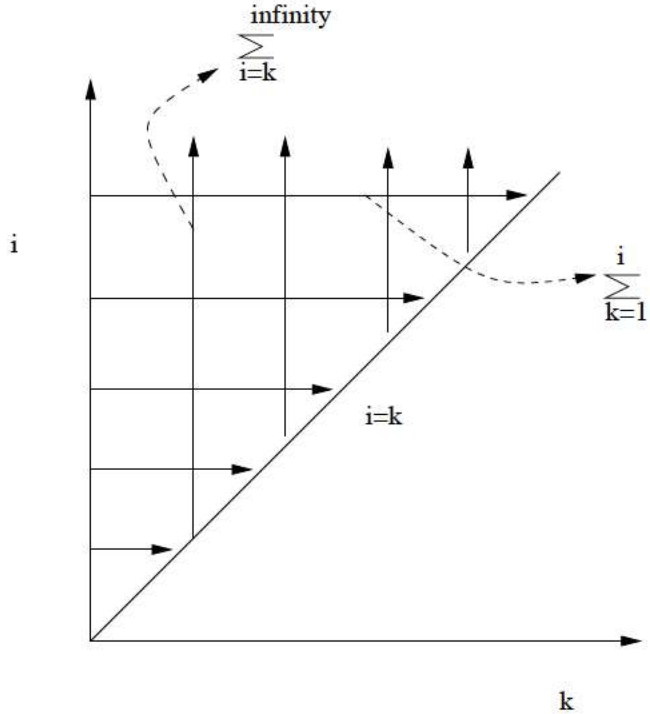
Solution: the probability of a given person picking up a clean hat is $1-p$. By the independence assumption, the probability of m selected persons picking up clean hats is $(1-p)^m$. (e) exactly m persons will pick up clean hats?

Answer: $(1-p)^m p^{n-m}\binom{n}{m}$.

Solution: every group G of m persons defines the event "everyone from G picks up a clean hat, everyone not from G picks up a dirty hat". The events are disjoint. Each has probability $(1-p)^m p^{n-m}$. Since there are $\binom{n}{m}$ such events, the answer follows.

16. Since 4 cards are fixed, Bob can only choose 4 more cards out of 48 remaining cards, so total number of hands Bob can have such that they include Alice's cards is $\binom{4}{4}\binom{48}{4}$. The total number of ways Bob can choose any 8 cards is $\binom{52}{8}$. So the probability is $\frac{\binom{4}{4}\binom{48}{4}}{\binom{52}{8}}$

17. (a) The picture below illustrates the double sum needed to prove the statement of this problem:



We first note that

$$\mathbf{P}(X \geq k) = \sum_{i=k}^{\infty} p_X(i)$$

and proceed as follows:

$$\sum_{k=1}^{\infty} \mathbf{P}(X \geq k) = \sum_{k=1}^{\infty}\sum_{i=k}^{\infty} p_X(i) = \sum_{i=1}^{\infty}\sum_{k=1}^{i} p_X(i) = \sum_{i=1}^{\infty} i p_X(i) = \mathbf{E}[X]$$

(b) We first compute

$$\mathbf{P}(Y \geq k) = \begin{cases} 1 & k \leq a \\ b-k+1 & a+1 \leq k \leq b \\ b-a+1 & k \geq b+1 \end{cases}$$

So

$$\begin{aligned}
\sum_{k=1}^{\infty} \mathbf{P}(Y \geq k) &= \sum_{k=1}^{a} 1 + \sum_{k=a+1}^{b} \frac{b-k+1}{b-a+1} \\
&= a + \frac{1}{b-a+1}\sum_{k=1}^{b-a} k \\
&= a + \frac{1}{b-a+1}\frac{(b-a+1)(b-a)}{2} \\
&= a + \frac{b-a}{2} \\
&= \frac{b+a}{2}
\end{aligned}$$

Therefore $\mathbf{E}[Y] = \frac{b+a}{2}$.

18. (a) For each value of X, we count the number of outcomes which have a difference that equals that value:

$$p_X(x) = \begin{cases} 1/9 & x = -2, 2 \\ 2/9 & x = -1, 1 \\ 3/9 & x = 0 \\ 0 & \text{otherwise.} \end{cases}$$

$$\mathrm{E}[X] = \sum_{x=-2}^{2} x p_X(x) = -2\frac{1}{9} + -1\frac{2}{9} + 0\frac{3}{9} + 1\frac{2}{9} + 2\frac{1}{9} = 0.$$

We can also see that $\mathbf{E}[X] = 0$ because the PMF is symmetric around 0 . To find the variance of X, we first compute

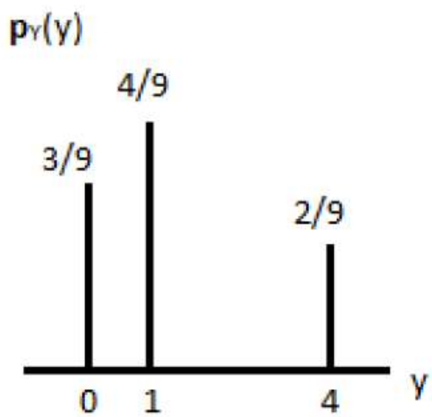
$$\mathrm{E}\left[X^2\right] = \sum_{x=-2}^{2} x^2 p_X(x) = 4\frac{1}{9} + 1\frac{2}{9} + 0\frac{3}{9} + 1\frac{2}{9} + 4\frac{1}{9} = \frac{4}{3}$$

and

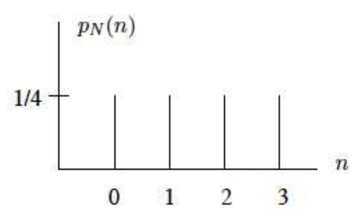
$$\operatorname{var}(X) = \mathbf{E}\left[X^2\right] - (\mathbf{E}[X])^2 = \frac{4}{3}.$$

(b) Let $Z = X^2$. By matching the possible values of X and their probabilities to the possible values of Z, we obtain

$$p_Z(z) = \begin{cases} 2/9 & z = 4 \\ 4/9 & z = 1 \\ 3/9 & z = 0 \\ 0 & \text{otherwise} \end{cases}$$



19. (a) The first part can be completed without reference to anything other than the die roll:



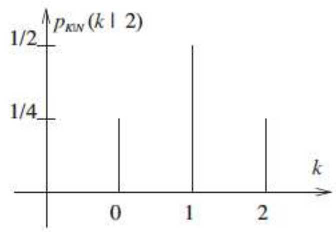
(b) When $N = 0$, the coin is not flipped at all, so $K = 0$. When $N = n$ for $n \in \{1, 2, 3\}$, the coin is flipped n times, resulting in K with a distribution that is conditionally binomial. The binomial probabilities are all multiplied by 1/4 because $p_N(n) = 1/4$ for $n \in \{0, 1, 2, 3\}$. The joint PMF $p_{N,K}(n, k)$ thus takes the following values and is zero otherwise:

	$k = 0$	$k = 1$	$k = 2$	$k = 3$
$n = 0$	1/4	0	0	0
$n = 1$	1/8	1/8	0	0
$n = 2$	1/16	1/8	1/16	0
$n = 3$	1/32	3/32	3/32	1/32

(c) Conditional on $N = 2, K$ is a binomial random variable. So we immediately see that

$$p_{K|N}(k \mid 2) = \begin{cases} 1/4, \text{ if } k = 0 \\ 1/2, \text{ if } k = 1 \\ 1/4, \text{ if } k = 2 \\ \quad 0, \text{ otherwise} \end{cases}$$

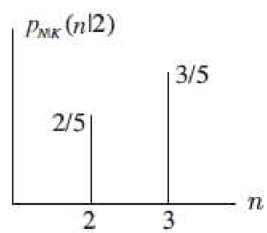
This is a normalized row of the table in the previous part.



(d) To get $K = 2$ heads, there must have been at least 3 coin tosses, so only $N = 3$ and $N = 4$ have positive conditional probability given $K = 2$.

$$p_{N|K}(2 \mid 2) = \frac{\mathbf{P}(\{N = 2\} \cap \{K = 2\})}{\mathbf{P}(\{K = 2\})} = \frac{1/16}{1/16 + 1/32 + 1/32 + 1/32} = 2/5.$$

Similarly, $p_{N|K}(3 \mid 2) = 3/5$.



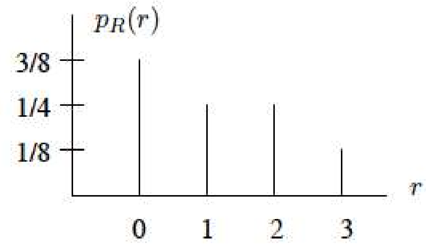
20. (a) $x = 0$ maximizes $\mathbf{E}[Y \mid X = x]$ since

$$\mathbf{E}[Y \mid X = x] = \begin{cases} 2, & \text{if } x = 0 \\ 3/2, & \text{if } x = 2 \\ 3/2, & \text{if } x = 4 \\ \text{undefined}, & \text{otherwise.} \end{cases}$$

(b) $y = 3$ maximizes $\text{var}(X \mid Y = y)$ since

$$\text{var}(X \mid Y = y) = \begin{cases} 0, & \text{if } y = 0 \\ 8/3, & \text{if } y = 1 \\ 1, & \text{if } y = 2 \\ 4, & \text{if } y = 3 \\ \text{undefined}, & \text{otherwise} \end{cases}$$

(c) Sketch:



(d) By traversing the points top to bottom and left to right, we obtain

$$\mathbf{E}[XY] = \frac{1}{8}(0 \cdot 3 + 4 \cdot 3 + 2 \cdot 2 + 4 \cdot 2 + 0 \cdot 1 + 2 \cdot 1 + 4 \cdot 1 + 4 \cdot 0) = \frac{15}{4}$$

Conditioning on A removes the point masses at $(0, 1)$ and $(0, 3)$. The conditional probability of each of the remaining point masses is thus $1/6$, and

$$\mathbf{E}[XY \mid A] = \frac{1}{6}(4 \cdot 3 + 2 \cdot 2 + 4 \cdot 2 + 2 \cdot 1 + 4 \cdot 1 + 4 \cdot 0) = 5$$

21. In general we have that $\mathbf{E}[aX + bY + c] = a\mathbf{E}[X] + b\mathbf{E}[Y] + c$. Therefore,

$$\mathbf{E}[Z] = 2 \cdot \mathbf{E}[X] - 3 \cdot \mathbf{E}[Y]$$

For the case of independent random variables, we have that if $Z = a \cdot X + b \cdot Y$, then

$$\text{var}(Z) = a^2 \cdot \text{var}(X) + b^2 \cdot \text{var}(Y)$$

Therefore, $\text{var}(Z) = 4 \cdot \text{var}(X) + 9 \cdot \text{var}(Y)$.

22. (a) We can find c knowing that the probability of the entire sample space must equal 1.

$$\begin{aligned} 1 &= \sum_{x=1}^{3} \sum_{y=1}^{3} p_{X,Y}(x, y) \\ &= c + c + 2c + 2c + 4c + 3c + c + 6c \\ &= 20c \end{aligned}$$

Therefore, $c = \frac{1}{20}$.

(b) $p_Y(2) = \sum_{x=1}^{3} p_{X,Y}(x, 2) = 2c + 0 + 4c = 6c = \frac{3}{10}$.

(c) $Z = YX^2$

$$\begin{aligned} \mathbf{E}[Z \mid Y = 2] &= \mathbf{E}\left[YX^2 \mid Y = 2\right] \\ &= \mathbf{E}\left[2X^2 \mid Y = 2\right] \\ &= 2\mathbf{E}\left[X^2 \mid Y = 2\right] \end{aligned}$$

$$p_{X|Y}(x \mid 2) = \frac{p_{X,Y}(x, 2)}{p_Y(2)}$$

Therefore,

$$p_{X|Y}(x \mid 2) = \begin{cases} \frac{1/10}{3/10} = \frac{1}{3} & \text{if } x = 1 \\ \frac{1/5}{3/10} = \frac{2}{3} & \text{if } x = 3 \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned} \mathbf{E}[Z \mid Y = 2] &= 2 \sum_{x=1}^{3} x^2 p_{X|Y}(x \mid 2) \\ &= 2\left(\left(1^2\right) \cdot \frac{1}{3} + \left(3^2\right) \cdot \frac{2}{3}\right) \\ &= \frac{38}{3} \end{aligned}$$

(d) Yes. Given $X \neq 2$, the distribution of X is the same given $Y = y$.

$$\mathbf{P}(X = x \mid Y = y, X \neq 2) = \mathbf{P}(X = x \mid X \neq 2).$$

For example,

$$\mathbf{P}(X = 1 \mid Y = 1, X \neq 2) = \mathbf{P}(X = 1 \mid Y = 3, X \neq 2) = \mathbf{P}(X = 1 \mid X \neq 2) = \frac{1}{3}$$

(e) $p_{Y|X}(y \mid 2) = \frac{p_{X,Y}(2,y)}{p_X(2)}$.

$$p_X(2) = \sum_{y=1}^{3} p_{X,Y}(2, y) = c + 0 + c = 2c = \frac{1}{10}.$$

Therefore,

$$p_{Y|X}(y \mid 2) = \begin{cases} \frac{1/20}{1/10} = \frac{1}{2} & \text{if } y = 1 \\ \frac{1/20}{1/10} = \frac{1}{2} & \text{if } y = 3 \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned} \mathbf{E}\left[Y^2 \mid X = 2\right] &= \sum_{y=1}^{3} y^2 p_{Y|X}(y \mid 2) = \left(1^2\right) \cdot \frac{1}{2} + \left(3^2\right) \cdot \frac{1}{2} = 5 \\ \mathbf{E}[Y \mid X = 2] &= \sum_{y=1}^{3} y p_{Y|X}(y \mid 2) = 1 \cdot \frac{1}{2} + 3 \cdot \frac{1}{2} = 2 \\ \operatorname{var}(Y \mid X = 2) &= \mathbf{E}\left[Y^2 \mid X = 2\right] - \mathbf{E}[Y \mid X = 2]^2 = 5 - 2^2 = 1 \end{aligned}$$

23. (a)

$$\begin{aligned} p_X(1) &= \mathbf{P}(X = 1, Y = 1) + \mathbf{P}(X = 1, Y = 2) + \mathbf{P}(X = 1, Y = 3) \\ &= 1/12 + 2/12 + 1/12 = 1/3 \end{aligned}$$

(b) The solution is a sketch of the following conditional PMF:

$$p_{Y|X}(y \mid 1) = \frac{p_{Y,X}(y,1)}{p_X(1)} = \begin{cases} 1/4, & \text{if } y = 1 \\ 1/2, & \text{if } y = 2 \\ 1/4, & \text{if } y = 3 \\ 0, & \text{otherwise} \end{cases}$$

(c) $\mathbf{E}[Y \mid X = 1] = \sum_{y=1}^{3} y p_{Y|X}(y \mid 1) = 1 \cdot \frac{1}{4} + 2 \cdot \frac{1}{2} + 3 \cdot \frac{1}{4} = 2$

(d) Assume that X and Y are independent. Because $p_{X,Y}(3,1) = 0$ and $p_Y(1) = 1/4, p_X(3)$ must equal zero. This further implies $p_{X,Y}(3,2) = 0$ and $p_{X,Y}(3,3) = 0$. All the remaining probability mass must go to $(X,Y) = (2,2)$, making $p_{X,Y}(2,2) = 5/12, p_X(2) = 8/12$, and $p_Y(2) = 7/12$. However, $p_{X,Y}(2,2) \neq p_X(2) \cdot p_Y(2)$, contradicting the assumption; thus X and Y are not independent. A simpler explanation uses only two X values and two Y values for which all four (X,Y) pairs have specified probabilities. Note that if X and Y are independent, then $p_{X,Y}(1,3)/p_{X,Y}(1,1)$ and $p_{X,Y}(2,3)/p_{X,Y}(2,1)$ must be equal because they must both equal $p_Y(3)/p_Y(1)$. This necessary equality does not hold, so X and Y are not independent.

(e) Knowing that X and Y are conditionally independent given B, we must have

$$\frac{p_{X,Y}(1,1)}{p_{X,Y}(1,2)} = \frac{p_{X,Y}(2,1)}{p_{X,Y}(2,2)}$$

since the (X,Y) pairs in the equality are all in B. Thus

$$p_{X,Y}(2,2) = \frac{p_{X,Y}(1,2) p_{X,Y}(2,1)}{p_{X,Y}(1,1)} = \frac{(2/12)(2/12)}{1/12} = \frac{4}{12} = \frac{1}{3}$$

(f) Since $\mathbf{P}(B) = 9/12 = 3/4$, we normalize to obtain $p_{X,Y|B}(2,2) = \frac{p_{X,Y}(2,2)}{\mathbf{P}(B)} = 4/9$.

24. (a) From the joint PMF, there are six (x, y) coordinate pairs with nonzero probabilities of occurring. These pairs are $(1,1),(1,3),(2,1),(2,3),(4,1)$, and $(4,3)$. The probability of a pair is proportional to the sum of the squares of the coordinates of the pair, x^2+y^2. Because the probability of the entire sample space must equal 1 , we have:

$$(1+1)c+(1+9)c+(4+1)c+(4+9)c+(16+1)c+(16+9)c=1$$

Solving for c, we get $c=\frac{1}{72}$.

(b) There are three sample points for which $y<x$:

$$\mathbf{P}(Y<X)=\mathbf{P}(\{(2,1)\})+\mathbf{P}(\{(4,1)\})+\mathbf{P}(\{(4,3)\})=\frac{5}{72}+\frac{17}{72}+\frac{25}{72}=\frac{47}{72}.$$

(c) There are two sample points for which $y>x$:

$$\mathbf{P}(Y>X)=\mathbf{P}(\{(1,3)\})+\mathbf{P}(\{(2,3)\})=\frac{10}{72}+\frac{13}{72}=\frac{23}{72}.$$

(d) There is only one sample point for which $y=x$:

$$\mathbf{P}(Y=X)=\mathbf{P}(\{(1,1)\})=\frac{2}{72}.$$

Notice that, using the above two parts,

$$\mathbf{P}(Y<X)+\mathbf{P}(Y>X)+\mathbf{P}(Y=X)=\frac{47}{72}+\frac{23}{72}+\frac{2}{72}=1$$

as expected.

(e) There are three sample points for which $y=3$:

$$\mathbf{P}(Y=3)=\mathbf{P}(\{(1,3)\})+\mathbf{P}(\{(2,3)\})+\mathbf{P}(\{(4,3)\})=\frac{10}{72}+\frac{13}{72}+\frac{25}{72}=\frac{48}{72}.$$

(f) In general, for two discrete random variable X and Y for which a joint PMF is defined, we have

$$p_X(x)=\sum_{y=-\infty}^{\infty} p_{X,Y}(x,y) \quad \text{and} \quad p_Y(y)=\sum_{x=-\infty}^{\infty} p_{X,Y}(x,y)$$

In this problem the ranges of X and Y are quite restricted so we can determine the marginal PMFs by enumeration. For example,

$$p_X(2)=\mathbf{P}(\{(2,1)\})+\mathbf{P}(\{(2,3)\})=\frac{18}{72}.$$

Overall, we get:

$$p_X(x)=\begin{cases} 12/72, & \text{if } x=1, \\ 18/72, & \text{if } x=2, \\ 42/72, & \text{if } x=4, \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad p_Y(y)=\begin{cases} 24/72, & \text{if } y=1 \\ 48/72, & \text{if } y=3 \\ 0, & \text{otherwise} \end{cases}$$

(g) In general, the expected value of any discrete random variable X equals

$$\mathbf{E}[X]=\sum_{x=-\infty}^{\infty} x p_X(x)$$

For this problem,

$$\mathbf{E}[X]=1\cdot\frac{12}{72}+2\cdot\frac{18}{72}+4\cdot\frac{42}{72}=3$$

and

$$\mathbf{E}[Y]=1\cdot\frac{24}{72}+3\cdot\frac{48}{72}=\frac{7}{3}.$$

To compute $\mathbf{E}[XY]$, note that $p_{X,Y}(x,y)\neq p_X(x)p_Y(y)$. Therefore, X and Y are not independent and we cannot assume $\mathbf{E}[XY]=\mathbf{E}[X]\mathbf{E}[Y]$. Thus, we have

$$\begin{aligned}\mathbf{E}[XY]&=\sum_x\sum_y xyp_{X,Y}(x,y)\\&=1\cdot\frac{2}{72}+2\cdot\frac{5}{72}+4\cdot\frac{17}{72}+3\cdot\frac{10}{72}+6\cdot\frac{13}{72}+12\cdot\frac{25}{72}=\frac{61}{9}.\end{aligned}$$

(h) The variance of a random variable X can be computed as $\mathbf{E}\left[X^2\right]-\mathbf{E}[X]^2$ or as $\mathbf{E}\left[(X-\mathbf{E}[X])^2\right]$. We use the second approach here because X and Y take on such limited ranges. We have

$$\operatorname{var}(X)=(1-3)^2\frac{12}{72}+(2-3)^2\frac{18}{72}+(4-3)^2\frac{42}{72}=\frac{3}{2}$$

and

$$\operatorname{var}(Y)=\left(1-\frac{7}{3}\right)^2\frac{24}{72}+\left(3-\frac{7}{3}\right)^2\frac{48}{72}=\frac{8}{9}.$$

X and Y are not independent, so we cannot assume $\operatorname{var}(X+Y)=\operatorname{var}(X)+\operatorname{var}(Y)$. The variance of $X+Y$ will be computed using $\operatorname{var}(X+Y)=\mathbf{E}\left[(X+Y)^2\right]-(\mathbf{E}[X+Y])^2$. Therefore, we have

$$\begin{aligned}\mathbf{E}\left[(X+Y)^2\right]&=4\cdot\frac{2}{72}+9\cdot\frac{5}{72}+25\cdot\frac{17}{72}+16\cdot\frac{10}{72}+25\cdot\frac{13}{72}+49\cdot\frac{25}{72}=\frac{547}{18}\\(\mathbf{E}[X+Y])^2&=(\mathbf{E}[X]+\mathbf{E}[Y])^2=\left(3+\frac{7}{3}\right)^2=\frac{256}{9}\end{aligned}$$

Therefore,

$$\operatorname{var}(X+Y)=\frac{547}{18}-\frac{256}{9}=\frac{35}{18}.$$

(i) There are four (x,y) coordinate pairs in $A:(1,1),(2,1),(4,1)$, and $(4,3)$. Therefore, $\mathbf{P}(A)=\frac{1}{72}(2+5+17+25)=\frac{49}{72}$. To find $\mathbf{E}[X\mid A]$ and $\operatorname{var}(X\mid A), p_{X\mid A}(x)$ must be calculated. We have

$$p_{X\mid A}(x)=\begin{cases}2/49, & \text{if } x=1\\5/49, & \text{if } x=2\\42/49, & \text{if } x=4\\0, & \text{otherwise}\end{cases}$$

$$\begin{aligned}\mathbf{E}[X\mid A]&=1\cdot\frac{2}{49}+2\cdot\frac{5}{49}+4\cdot\frac{42}{49}=\frac{180}{49}\\\mathbf{E}\left[X^2\mid A\right]&=1^2\cdot\frac{2}{49}+2^2\cdot\frac{5}{49}+4^2\cdot\frac{42}{49}=\frac{694}{49}\\\operatorname{var}(X\mid A)&=\mathbf{E}\left[X^2\mid A\right]-(\mathbf{E}[X\mid A])^2=\frac{694}{49}-\left(\frac{180}{49}\right)^2=\frac{1606}{2401}\end{aligned}$$

25. Consider a sequence of six independent rolls of this die, and let X_i be the random variable corresponding to the i th roll.

(a) What is the probability that exactly three of the rolls have result equal to 3 ? Each roll X_i can either be a 3 with probability $1/4$ or not a 3 with probability $3/4$. There are $\binom{6}{3}$ ways of placing the 3's in the sequence of six rolls. After we require that a 3 go in each of these spots, which has probability $(1/4)^3$, our only remaining condition is that either a 1 or a 2 go in the other three spots, which has probability $(3/4)^3$. So the probability of exactly three rolls of 3 in a sequence of six independent rolls is $\binom{6}{3}\left(\frac{1}{4}\right)^3\left(\frac{3}{4}\right)^3$.

(b) What is the probability that the first roll is 1 , given that exactly two of the six rolls have result of 1 ? The probability of obtaining a 1 on a single roll is 1/2, and the probability of obtaining a 2 or 3 on a single roll is also 1/2. For the purposes of solving this problem we treat obtaining a 2 or 3 as an equivalent result. We know that there are $\binom{6}{2}$ ways of rolling exactly two 1's. Of these $\binom{6}{2}$ ways, exactly $\binom{5}{1}=5$ ways result in a 1 in the first roll, since we can place the remaining 1 in any of the five remaining rolls. The rest of the rolls must be either 2 or 3 . Thus, the probability that the first roll is a 1 given exactly two rolls had an outcome of 1 is $\frac{5}{\binom{6}{2}}$.

(c) We are now told that exactly three of the rolls resulted in 1 and exactly three resulted in 2. What is the probability of the sequence 121212 ? We want to find

$$\mathbf{P}(121212 \mid \text{exactly three 1's and three 2's }) = \frac{\mathbf{P}(121212)}{\mathbf{P}(\text{ exactly 3 ones and 3 twos })}.$$

Any particular sequence of three 1's and three 2 's will have the same probability: $(1/2)^3(1/4)^3$. There are $\binom{6}{3}$ possible rolls with exactly three 1's and three 2 's. Therefore,

$$\mathbf{P}(121212 | \text{exactly three 1's and three 2's }) = \frac{1}{\binom{6}{3}}$$

(d) Conditioned on the event that at least one roll resulted in 3, find the conditional PMF of the number of 3's. Let A be the event that at least one roll results in a 3. Then

$$\mathbf{P}(A) = 1 - \mathbf{P}(\text{ no rolls resulted in 3}) = 1 - \left(\frac{3}{4}\right)^6$$

Now let K be the random variable representing the number of 3 's in the 6 rolls. The (unconditional) PMF $p_K(k)$ for K is given by

$$p_K(k) = \binom{6}{k}\left(\frac{1}{4}\right)^k\left(\frac{3}{4}\right)^{6-k}$$

We find the conditional PMF $p_{k|A}(k \mid A)$ using the definition of conditional probability:

$$p_{K|A}(k \mid A) = \frac{\mathbf{P}(\{K=k\}\cap A)}{\mathbf{P}(A)}.$$

Thus we obtain

$$p_{K|A}(k \mid A) = \begin{cases} \frac{1}{1-(3/4)^6}\binom{6}{k}\left(\frac{1}{4}\right)^k\left(\frac{3}{4}\right)^{6-k} & \text{if } k = 1,2,\ldots,6, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $p_{K|A}(0 \mid A) = 0$ because the event $\{K=0\}$ and the event A are mutually exclusive. Thus the probability of their intersection, which appears in the numerator in the definition of the conditional PMF, is zero.

26. By the definition of conditional probability,

$$\mathbf{P}(X=i \mid X+Y=n) = \frac{\mathbf{P}(\{X=i\}\cap\{X+Y=n\})}{\mathbf{P}(X+Y=n)}.$$

The event $\{X=i\}\cap\{X+Y=n\}$ in the numerator is equivalent to $\{X=i\}\cap\{Y=n-i\}$. Combining this with the independence of X and Y,

$$\mathbf{P}(\{X=i\}\cap\{X+Y=n\}) = \mathbf{P}(\{X=i\}\cap\{Y=n-i\}) = \mathbf{P}(X=i)\mathbf{P}(Y=n-i).$$

In the denominator, $\mathbf{P}(X+Y=n)$ can be expanded using the total probability theorem and the independence of X and Y :

$$\begin{aligned}
\mathbf{P}(X+Y=n) &= \sum_{i=1}^{n-1} \mathbf{P}(X=i)\mathbf{P}(X+Y=n \mid X=i) \\
&= \sum_{i=1}^{n-1} \mathbf{P}(X=i)\mathbf{P}(i+Y=n \mid X=i) \\
&= \sum_{i=1}^{n-1} \mathbf{P}(X=i)\mathbf{P}(Y=n-i \mid X=i) \\
&= \sum_{i=1}^{n-1} \mathbf{P}(X=i)\mathbf{P}(Y=n-i)
\end{aligned}$$

Note that we only get non-zero probability for $i = 1, \ldots, n-1$ since X and Y are geometric random variables. The desired result is obtained by combining the computations above and using the geometric PMF explicitly:

$$\begin{aligned}
\mathbf{P}(X=i \mid X+Y=n) &= \frac{\mathbf{P}(X=i)\mathbf{P}(Y=n-i)}{\sum_{i=1}^{n-1}\mathbf{P}(X=i)\mathbf{P}(Y=n-i)} \\
&= \frac{(1-p)^{i-1}p(1-p)^{n-i-1}p}{\sum_{i=1}^{n-1}(1-p)^{i-1}p(1-p)^{n-i-1}p} \\
&= \frac{(1-p)^n}{\sum_{i=1}^{n-1}(1-p)^n} \\
&= \frac{(1-p)^n}{(1-p)^n\sum_{i=1}^{n-1}1} \\
&= \frac{1}{n-1}, \quad i = 1, \ldots, n-1
\end{aligned}$$

27. (a) Let R_i be the amount of time Stephen spends at the i th red light. R_i is a Bernoulli random variable with $p = 1/3$. The PMF for R_i is:

$$\mathbf{P}_{R_i}(r) = \begin{cases} 2/3, & \text{if } r = 0 \\ 1/3, & \text{if } r = 1 \\ 0, & \text{otherwise.} \end{cases}$$

The expectation and variance for R_i are:

$$\begin{aligned}
\mathbf{E}[R_i] &= p = \frac{1}{3} \\
\operatorname{var}(R_i) &= p(1-p) = \frac{1}{3}\frac{2}{3} = \frac{2}{9}
\end{aligned}$$

Let T_S be the total length of time of Stephen's commute in minutes. Then,

$$T_S = 18 + \sum_{i=1}^{5} R_i$$

T_S is a shifted binomial with $n = 5$ trials and $p = 1/3$. The PMF for T_S is then:

$$\mathbf{P}_{T_S}(k) = \begin{cases} \binom{5}{k-18}\left(\frac{1}{3}\right)^{k-18}\left(\frac{2}{3}\right)^{23-k}, & \text{if } k \in \{18, 19, 20, 21, 22, 23\} \\ 0, & \text{otherwise.} \end{cases}$$

The expectation and variance for T_S are:

$$
\begin{aligned}
\mathbf{E}\left[T_S\right] &= \mathbf{E}\left[18+\sum_{i=1}^{5} R_i\right] \\
&= \frac{59}{3} \\
\operatorname{var}\left(T_S\right) &= \operatorname{var}\left(18+\sum_{i=1}^{5} R_i\right) \\
&= \frac{10}{9}
\end{aligned}
$$

(b) Let N be the number of red lights Stephen encountered on his commute. Given that $T_S \leq 19$, then $N=0$ or $N=1$. The unconditional probability of $N=0$ is $\mathbf{P}(N=0)=\left(\frac{2}{3}\right)^5$. The unconditional probability of $N=1$ is $\mathbf{P}(N=1)=\binom{5}{1}\left(\frac{2}{3}\right)^4\left(\frac{1}{3}\right)^1$. To find the conditional expectation, the following conditional PDF is calculated:

$$
\mathbf{P}_{N \mid T_S \leq 19}\left(n \mid T_S \leq 19\right)=\begin{cases}\frac{\left(\frac{2}{3}\right)^5}{\left(\frac{2}{3}\right)^5+\binom{5}{1}\left(\frac{2}{3}\right)^4\left(\frac{1}{3}\right)^1}, & \text{if } n=0, \\ \frac{\binom{5}{1}\left(\frac{2}{3}\right)^4\left(\frac{1}{3}\right)^1}{\left(\frac{2}{3}\right)^5+\binom{5}{1}\left(\frac{2}{3}\right)^4\left(\frac{1}{3}\right)^1}, & \text{if } n=1, \\ 0, & \text{otherwise,}\end{cases} \quad = \begin{cases}2/7, & \text{if } n=0, \\ 5/7, & \text{if } n=1, \\ 0, & \text{otherwise.}\end{cases}
$$

Therefore,

$$
\mathbf{E}\left[N \mid T_S \leq 19\right]=\frac{5}{7}
$$

(c) Given that the last red light encountered by Stephen was the fourth light, $R_4=1$ and $R_5=0$. We are asked to compute $\operatorname{var}\left(N \mid \{R_4=1\} \cap \{R_5=0\}\right)$. Therefore,

$$
\begin{aligned}
\operatorname{var}\left(N \mid \{R_4=1\} \cap \{R_5=0\}\right) &= \operatorname{var}\left(R_1+R_2+R_3+R_4+R_5 \mid \{R_4=1\} \cap \{R_5=0\}\right) \\
&= \operatorname{var}\left(R_1+R_2+R_3+1+0 \mid \{R_4=1\} \cap \{R_5=0\}\right) \\
&= \operatorname{var}\left(R_1+R_2+R_3+1\right) \\
&= \operatorname{var}\left(R_1+R_2+R_3\right) \\
&= 3 \operatorname{var}\left(R_1\right) \\
&= \frac{6}{9}.
\end{aligned}
$$

(d) Under the given condition, the discrete uniform law can be used to compute the probability of interest. There are $\binom{5}{3}$ ways that Stephen can encounter a total of three red lights. There are $\binom{3}{2}$ ways that two out of the first three lights were red. This leaves one additional red light out of the last two lights and there are $\binom{2}{1}$ possible ways that this event can occur. Putting it all together,

$$
\mathbf{P}\left(\text{two of first three lights were red} \mid \text{total of three red lights}\right)=\frac{\binom{3}{2}\binom{2}{1}}{\binom{5}{3}}=\frac{3}{5}.
$$

(e) Let T_J be the total length of time of Jon's commute in minutes. The PMF of Jon's commute is:

$$
\mathbf{P}_{T_J}(\ell)=\begin{cases}\frac{1}{4}, & \text{if } \ell \in \{20,21,22,23\} \\ 0, & \text{otherwise.}\end{cases}
$$

(f) Let A be the event that Jon arrives at work in 20 minutes and let B be the event that exactly one person arrives in 20 minutes.

$$
\begin{aligned}
\mathbf{P}(A \mid B) &= \frac{\mathbf{P}(A \cap B)}{\mathbf{P}(B)} \\
&= \frac{\mathbf{P}\left(\{T_J = 20\} \cap \{T_S \neq 20\}\right)}{\mathbf{P}\left(\{T_J = 20\} \cap \{T_S \neq 20\}\right) + \mathbf{P}\left(\{T_J \neq 20\} \cap \{T_S = 20\}\right)} \\
&= \frac{\mathbf{P}\left(T_J = 20\right) \mathbf{P}\left(T_S \neq 20\right)}{\mathbf{P}\left(T_J = 20\right) \mathbf{P}\left(T_S \neq 20\right) + \mathbf{P}\left(T_J \neq 20\right) \mathbf{P}\left(T_S = 20\right)}.
\end{aligned}
$$

Jon arrives at work in 20 minutes (or $T_J = 20$) if he does not have to wait for the train at the station (or $X = 0$). The probability of this event occurring is:

$$
\mathbf{P}\left(T_J = 20\right) = \mathbf{P}(X = 0) = \frac{1}{4}
$$

Stephen arrives at work in 20 minutes if he encounters 2 red lights. The probability of this event is a binomial probability:

$$
\mathbf{P}\left(T_S = 20\right) = \binom{5}{2} \left(\frac{1}{3}\right)^2 \left(\frac{2}{3}\right)^3
$$

Thus,

$$
\mathbf{P}(A \mid B) = \frac{\frac{1}{4}\left(1 - \binom{5}{2}\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^3\right)}{\frac{1}{4}\left(1 - \binom{5}{2}\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^3\right) + \frac{3}{4}\left(\binom{5}{2}\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^3\right)}.
$$

(g) The probability of interest is $\mathbf{P}\left(T_S \leq T_J\right)$. This can be calculated using the total probability theorem by conditioning on the length of Jon's commute or Jon's wait at the station. If Jon's commute is 20 minutes (or $X = 0$), then Stephen can encounter up to 2 red lights to satisfy $T_S \leq T_J$. Similarly if Jon's commute is 21 minutes (or $X = 1$), Stephen can encounter up to 3 red lights and so on.

$$
\begin{aligned}
\mathbf{P}\left(T_S \leq T_J\right) &= \sum_{x=0}^{3} \mathbf{P}\left(T_S \leq T_J \mid X = x\right) \mathbf{P}(X = x) \\
&= \frac{1}{4} \sum_{x=0}^{3} \sum_{k=0}^{2+x} \binom{5}{k} \left(\frac{1}{3}\right)^k \left(\frac{2}{3}\right)^{5-k} \\
&= 0.9352
\end{aligned}
$$

An alternative approach follows. We first compute the joint PMF of the commute times of Stephen and Jon $\mathbf{P}_{T_S, T_J}(k, \ell)$. Because of independence, $\mathbf{P}_{T_S, T_J}(k, \ell) = \mathbf{P}_{T_S}(k) \mathbf{P}_{T_J}(\ell)$. Therefore,

$$
\begin{aligned}
\mathbf{P}\left(T_S \leq T_J\right) =& \mathbf{P}\left(T_S = 18\right) + \mathbf{P}\left(T_S = 19\right) + \mathbf{P}\left(T_S = 20\right) + \mathbf{P}\left(\{T_S = 21\} \cap \{T_J \geq 21\}\right) \\
&+ \mathbf{P}\left(\{T_S = 22\} \cap \{T_J \geq 22\}\right) + \mathbf{P}\left(\{T_S = 23\} \cap \{T_J = 23\}\right) \\
=& \left(\frac{2}{3}\right)^5 + \binom{5}{1}\left(\frac{1}{3}\right)^1\left(\frac{2}{3}\right)^4 + \binom{5}{2}\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^3 + \binom{5}{3}\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^2 \cdot \left(\frac{3}{4}\right) \\
&+ \binom{5}{4}\left(\frac{1}{3}\right)^4\left(\frac{2}{3}\right)^1 \cdot \left(\frac{2}{4}\right) + \left(\frac{1}{3}\right)^5 \cdot \left(\frac{1}{4}\right) \\
=& 0.9352.
\end{aligned}
$$

(h) We express the conditional probability as such:

$$
\mathbf{P}\left(X = 3 \mid T_S \leq T_J\right) = \frac{\mathbf{P}\left(\{X = 3\} \cap \{T_S \leq T_J\}\right)}{\mathbf{P}\left(T_S \leq T_J\right)}
$$

If Jon waited 3 minutes at the train, his commute was 23 minutes and Stephen's commute takes at most as long as Jon's commute since the longest possible commute for Stephen is 23 minutes. Therefore, the numerator in the previous expression is equal to $\mathbf{P}(X = 3) = \frac{1}{4}$. The denominator was computed in the previous part.

$$
\begin{aligned}
\mathbf{P}(X=3 \mid T_S \le T_J) &= \frac{1}{\sum_{x=0}^{3}\sum_{k=0}^{2+x}\binom{5}{k}\left(\frac{1}{3}\right)^k\left(\frac{2}{3}\right)^{5-k}} \\
&= 0.2673
\end{aligned}
$$

28. (a)

$$
p_X(x) = \sum_{y=1}^{4} p_{X,Y}(x,y) = \begin{cases} \frac{1}{4} & x = 1,2,3,4 \\ 0 & \text{o.w.} \end{cases}
$$

(b)

$$
p_Y(y) = \sum_{x=1}^{4} p_{X,Y}(x,y) = \begin{cases} \frac{1}{4} & y = 1,2,3,4 \\ 0 & \text{o.w.} \end{cases}
$$

(c) No. One of many counter examples: $p_X(x)$ does not equal $p_{X|Y}(x \mid 2)$.

29. (d) (continuation to the earlier question) Let event D be the set of all doubles, and let event A be the event that Alice's coin toss results in heads. Using the law of total probability:

$$
\begin{aligned}
P(D) &= P(D \mid A)P(A) + P(D \mid A^c)\,P(A^c) \\
&= \frac{3}{4} \times \frac{4}{10} + \frac{1}{4} \times \frac{1}{4} \\
&= \frac{58}{160} = .3625
\end{aligned}
$$

(e) Let random variable N be the number of rolls until doubles is rolled. The distribution on N condition on the set of dice being rolled is a geometric random variable. Using the total expectation theorem, the expected value of N is:

$$
\begin{aligned}
E[N] &= E[N \mid A]P(A) + E[N \mid A^c]\,P(A^c) \\
&= \frac{3}{4} \times \frac{1}{\frac{4}{10}} + \frac{1}{4} \times \frac{1}{\frac{1}{4}} \\
&= \frac{23}{8} = 2.875
\end{aligned}
$$

(f) The time T until Alice sees a total of 11 heads is the sum of 11 independent and identically distributed geometric random variables with parameter $p = \frac{3}{4}$. Random variable R, the number of tails she sees, is $T - 11$. Thus:

$$
\begin{aligned}
E[R] &= E[T] - 11 \\
&= 11 \times \frac{1}{\frac{3}{4}} - 11 \\
&= \frac{11}{12} = .9167
\end{aligned}
$$

(g) The probability of event A can be found by choosing one of the first 6 outcomes to be a head, the others tails, and then the outcome of the 7th toss to be head, which is $\binom{6}{1}(1-p)^5 p^2$, where $p = \frac{3}{4}$. The intersection of $S = s$ with event A, $P(S = s \cap A)$, is an event with probability $(1-p)^5 p^2$ for all values of $s (s = 1, \ldots, 6)$. Consequently, $p_{S|A}(s) = \frac{P(S=s\cap A)}{P(A)}$ is a uniform distribution over the range of $s (s = 1, \ldots, 6)$.

$$
p_{S|A}(s) = \begin{cases} \frac{1}{6} & s = 1, \ldots, 6 \\ 0 & o.w. \end{cases}
$$

(h) Bob must toss a coin at least 11 times and at most 21 times in order to have either 11 heads or 11 tails. The intersection of Bob requiring u tosses and 11 of those tosses being heads, is the sum of probability the $\binom{u-1}{10}$ sequences that conclude with a head and have a total of 11 heads. The probability of each of those sequences is $\left(\frac{1}{2}\right)^u$. If we consider any sequence in Bob's experiment with u tosses, since the coin is fair, that sequence is equally likely to have 11 heads and $u - 11$ tails or 11 tails and $u - 11$ heads. Consequently, the intersection of Bob requiring u tosses and 11 of those tosses being tails is identical to the probability that the sequence had 11 heads. Summing these two mutually exclusive probabilities which total $p_U(u)$:

$$p_U(u) = \begin{cases} 2\binom{u-1}{10}\left(\frac{1}{2}\right)^u & u = 11, \ldots, 21 \\ 0 & \text{o.w.} \end{cases}$$

30. (a) We know that the PDF must integrate to 1 . Therefore we have

$$\int_{-\infty}^{\infty} f_Z(z)dz = \int_{-2}^{1} \gamma\left(1+z^2\right) = \gamma\left(z + \frac{1}{3}z^3\right)\Big|_{-2}^{1} = 6\gamma$$

From this we conclude $\gamma = 1/6$.

(b) To find the CDF, we integrate:

$$F_Z(z) = \int_{-\infty}^{z} f_Z(t)dt = \begin{cases} 0, & \text{if } z < -2, \\ \frac{1}{6}\left(t + \frac{1}{3}t^3\right)\big|_{-2}^{z}, & \text{if } -2 \le z \le 1 \\ 1, & \text{if } z > 1 \end{cases}$$

$$= \begin{cases} 0, & \text{if } z < -2, \\ \frac{1}{6}\left(z + \frac{1}{3}z^3 + \frac{14}{3}\right), & \text{if } -2 \le z \le 1, \\ 1, & \text{if } z > 1 \end{cases}$$

31. (a)

$$\begin{aligned} \mathbf{P}(X \le 1.5) &= \Phi(1.5) \\ &= 0.9332 \end{aligned}$$

$$\begin{aligned} \mathbf{P}(X \le -1) &= 1 - \mathbf{P}(X \le 1) \\ &= 1 - \Phi(1) \\ &= 1 - 0.8413 \\ &= 0.1587. \end{aligned}$$

(b)

$$\begin{aligned} \mathbf{E}\left[\frac{Y-1}{2}\right] &= \frac{1}{2}(\mathbf{E}[Y] - 1) \\ &= 0 \\ \operatorname{var}\left(\frac{Y-1}{2}\right) &= \operatorname{var}\left(\frac{Y}{2}\right) \\ &= \frac{1}{4}\operatorname{var} Y \\ &= 1 \end{aligned}$$

Thus, the distribution of $\frac{Y-1}{2}$ is $\mathcal{N}(0,1)$.

(c)

$$\begin{aligned} \mathbf{P}(-1 \le Y \le 1) &= \mathbf{P}\left(\frac{-1-1}{2} \le \frac{Y-1}{2} \le \frac{1-1}{2}\right) \\ &= \Phi(0) - \Phi(-1) \\ &= \Phi(0) - (1 - \Phi(1)) \\ &= 0.3413 \end{aligned}$$

32. We first compute the probability that X is in interval $[n, n+1]$ for an arbitrary nonnegative n. Then, we will add the probabilities for all odd positive integer values of n. We could integrate the PDF of X over the given interval but we will use the CDF here. Using the CDF for the exponential random variable,

$$\begin{aligned} \mathbf{P}(n \le X \le n+1) &= F_X(n+1) - F_X(n) \\ &= \left(1 - e^{-\lambda(n+1)}\right) - \left(1 - e^{-\lambda n}\right) \\ &= e^{-\lambda n}\left(1 - e^{-\lambda}\right). \end{aligned}$$

Since the intervals are disjoint, we can sum this probability for all odd integers n to find the probability of interest:

$$\begin{aligned}
&\mathbf{P}(\{X \in [n, n+1] \text{ for some odd } n\}) \\
&= \sum_{n \text{ odd}} e^{-\lambda n}\left(1 - e^{-\lambda}\right) \\
&= \left(1 - e^{-\lambda}\right) \sum_{k=0}^{\infty} e^{-\lambda(2k+1)} \\
&= \left(1 - e^{-\lambda}\right) e^{-\lambda} \sum_{k=0}^{\infty} \left(e^{-2\lambda}\right)^k \\
&= \left(1 - e^{-\lambda}\right) e^{-\lambda} \frac{1}{1 - e^{-2\lambda}} \\
&= \left(1 - e^{-\lambda}\right) e^{-\lambda} \frac{1}{\left(1 - e^{-\lambda}\right)\left(1 + e^{-\lambda}\right)} \\
&= \frac{e^{-\lambda}}{1 + e^{-\lambda}}
\end{aligned}$$

33. Solution (a) Since

$$1 = \int_{-\infty}^{\infty} f(x)dx = \lambda \int_0^{\infty} e^{-x/100}dx$$

we obtain

$$1 = -\lambda(100)e^{-x/100}\Big|_0^{\infty} = 100\lambda \quad \text{or} \quad \lambda = \frac{1}{100}$$

Hence the probability that a computer will function between 50 and 150 hours before breaking down is given by

$$\begin{aligned}
P\{50 < X < 150\} &= \int_{50}^{150} \frac{1}{100} e^{-x/100}dx = -e^{-x/100}\Big|_{50}^{150} \\
&= e^{-1/2} - e^{-3/2} \approx .384
\end{aligned}$$

(b) Similarly,

$$P\{X < 100\} = \int_0^{100} \frac{1}{100} e^{-x/100}dx = -e^{-x/100}\Big|_0^{100} = 1 - e^{-1} \approx .633$$

In other words, approximately 63.3 percent of the time a computer will fail before registering 100 hours of use.

34. (a)

$$\begin{aligned}
P\{2 < X < 5\} &= P\left\{\frac{2-3}{3} < \frac{X-3}{3} < \frac{5-3}{3}\right\} = P\left\{-\frac{1}{3} < Z < \frac{2}{3}\right\} \\
&= \Phi\left(\frac{2}{3}\right) - \Phi\left(-\frac{1}{3}\right) \\
&= \Phi\left(\frac{2}{3}\right) - \left[1 - \Phi\left(\frac{1}{3}\right)\right] \approx .3779
\end{aligned}$$

(b)

$$\begin{aligned}
P\{X > 0\} = P\left\{\frac{X-3}{3} > \frac{0-3}{3}\right\} &= P\{Z > -1\} \\
&= 1 - \Phi(-1) \\
&= \Phi(1) \\
&\approx .8413
\end{aligned}$$

(c)

$$
\begin{aligned}
P\{|X-3|>6\} &= P\{X>9\}+P\{X<-3\} \\
&= P\left\{\frac{X-3}{3}>\frac{9-3}{3}\right\}+P\left\{\frac{X-3}{3}<\frac{-3-3}{3}\right\} \\
&= P\{Z>2\}+P[Z<-2\} \\
&= 1-\Phi(2)+\Phi(-2) \\
&= 2[1-\Phi(2)] \\
&\approx .0456
\end{aligned}
$$

35. Solution (a)

$$
\begin{aligned}
P\{X>1, Y<1\} &= \int_0^1 \int_1^\infty 2e^{-x}e^{-2y}dxdy \\
&= \int_0^1 2e^{-2y}\left(-\left.e^{-x}\right|_1^\infty\right)dy \\
&= e^{-1}\int_0^1 2e^{-2y}dy \\
&= e^{-1}\left(1-e^{-2}\right)
\end{aligned}
$$

(b)

$$
\begin{aligned}
P\{X<Y\} &= \iint_{(x,y):x<y} 2e^{-x}e^{-2y}dxdy \\
&= \int_0^x \int_0^y 2e^{-x}e^{-2y}dxdy \\
&= \int_0^x 2e^{-2y}\left(1-e^{-y}\right)dy \\
&= \int_0^\infty 2e^{-2y}dy - \int_0^\infty 2e^{-3y}dy \\
&= 1-\frac{2}{3} \\
&= \frac{1}{3}
\end{aligned}
$$

(c)

$$
\begin{aligned}
P\{X<a\} &= \int_0^a \int_0^\infty 2e^{-2y}e^{-x}dydx \\
&= \int_0^a e^{-x}dx \\
&= 1-e^{-a}
\end{aligned}
$$

36. For $0<x<1, 0<y<1$, we have

$$
\begin{aligned}
f_{X|Y}(x \mid y) &= \frac{f(x,y)}{f_Y(y)} \\
&= \frac{f(x,y)}{\int_{-\infty}^\infty f(x,y)dx} \\
&= \frac{x(2-x-y)}{\int_0^1 x(2-x-y)dx} \\
&= \frac{x(2-x-y)}{\frac{2}{3}-y/2} \\
&= \frac{6x(2-x-y)}{4-3y}
\end{aligned}
$$

37. We first obtain the conditional density of X, given that $Y = y$.

$$\begin{aligned} f_{X|Y}(x \mid y) &= \frac{f(x,y)}{f_Y(y)} \\ &= \frac{e^{-x/y}e^{-y}/y}{e^{-y}\int_0^x (1/y)e^{-x/y}dx} \\ &= \frac{1}{y}e^{-x/y} \end{aligned}$$

Hence

$$\begin{aligned} P\{X > 1 \mid Y = y\} &= \int_1^\infty \frac{1}{y}e^{-x/y}dx \\ &= -\left. e^{-x/y}\right|_1^\infty \\ &= e^{-1/y} \end{aligned}$$

38. (a) .9838 is found in the 2.1 row and the .04 column of the standard normal table so $c = 2.14$.

(b) $P(0 \le Z \le c) = .291 \Rightarrow \Phi(c) - \Phi(0) = .2910 \Rightarrow \Phi(c) - .5 = .2910 \Rightarrow \Phi(c) = .7910 \Rightarrow$ from the standard normal table, $c = .81$.

(c) $P(c \le Z) = .121 \Rightarrow 1 - P(Z < c) = .121 \Rightarrow 1 - \Phi(c) = .121 \Rightarrow \Phi(c) = .879 \Rightarrow c = 1.17$.

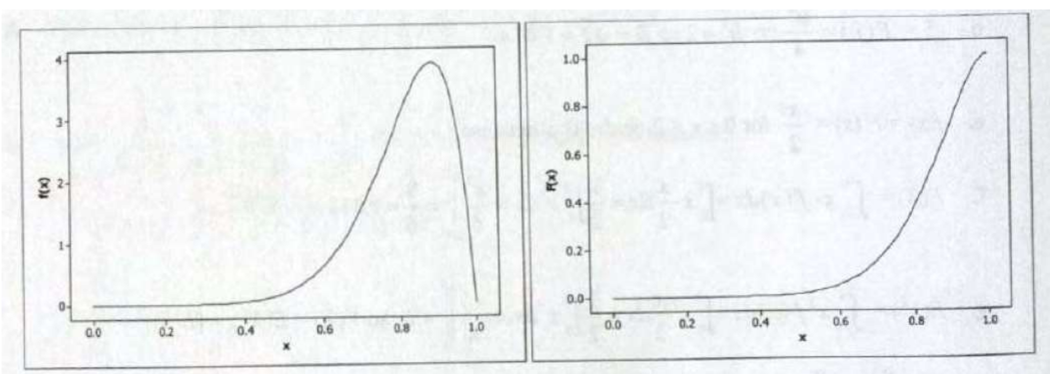
(d) $P(-c \le Z \le c) = \Phi(c) - \Phi(-c) = \Phi(c) - (1 - \Phi(c)) = 2\Phi(c) - 1 = .668 \Rightarrow \Phi(c) = .834 \Rightarrow c = 0.97$.

(e) $P(c \le |Z|) = 1 - P(|Z| < c) = 1 - [\Phi(c) - \Phi(-c)] = 1 - [2\Phi(c) - 1] = 2 - 2\Phi(c) = .016 \Rightarrow \Phi(c) = .992 \Rightarrow c = 2.41$.

39. (a) Since X is limited to the interval $(0, 1)$, $F(x) = 0$ for $x \le 0$ and $F(x) = 1$ for $x \ge 1$. For $0 < x < 1$,

$$F(x) = \int_{-\infty}^x f(y)dy = \int_0^x 90y^8(1-y)dy = \int_0^x \left(90y^8 - 90y^9\right)dy = 10y^9 - 9y^{10}\Big]_0^x = 10x^9 - 9x^{10}.$$

The graphs of the pdf and cdf of X appear below.



(b) $F(.5) = 10(.5)^9 - 9(.5)^{10} = .0107$.

(c) $P(.25 < X \le .5) = F(.5) - F(.25) = .0107 - \left[10(.25)^9 - 9(.25)^{10}\right] = .0107 - .0000 = .0107$. Since X is continuous, $P(.25 \le X \le .5) = P(.25 < X \le .5) = .0107$.

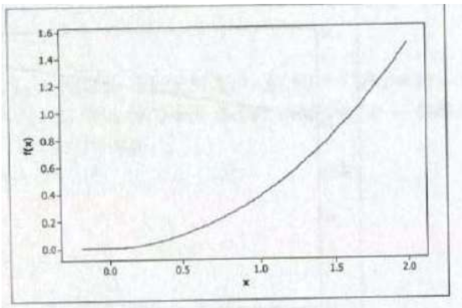
(d) The 75^{th} percentile is the value of x for which $F(x) = .75 : 10x^9 - 9x^{10} = .75 \Rightarrow x = .9036$ using software.

(e) $E(X) = \int_{-\infty}^{\infty} x \cdot f(x)dx = \int_0^1 x \cdot 90x^8(1-x)dx = \int_0^1 \left(90x^9 - 90x^{10}\right)dx = 9x^{10} - \frac{90}{11}x^{11}\Big]_0^1 = 9 - \frac{90}{11} = \frac{9}{11} = .8182$.

Similarly, $E\left(X^2\right) = \int_{-\infty}^{\infty} x^2 \cdot f(x)dx = \int_0^1 x^2 \cdot 90x^8(1-x)dx = \ldots = .6818$, from which $V(X) = .6818 - (.8182)^2 = .0124$ and $\sigma_X = .11134$.

(f) $\mu \pm \sigma = (.7068, .9295)$. Thus, $P(\mu - \sigma \le X \le \mu + \sigma) = F(.9295) - F(.7068) = .8465 - .1602 = .6863$, and the probability X is more than 1 standard deviation from its mean value equals $1 - .6863 = 3137$.

40. (a) $\quad 1=\int_{-\infty}^{\infty} f(x)dx=\int_0^2 kx^2dx=\frac{kx^3}{3}\Big]_0^2=\frac{8k}{3}\Rightarrow k=\frac{3}{8}.$



b. $\quad P(0\le X\le 1)=\int_0^1 \frac{3}{8}x^2dx=\frac{1}{8}x^3\Big]_0^1=\frac{1}{8}=.125.$

c. $\quad P(1\le X\le 1.5)=\int_1^{1.5} \frac{3}{8}x^2dx=\frac{1}{8}x^3\Big]_1^{15}=\frac{1}{8}\left(\frac{3}{2}\right)^3-\frac{1}{8}(1)^3=\frac{19}{64}=.296875.$

d. $\quad P(X\ge 1.5)=1-\int_{1.5}^2 \frac{3}{8}x^2dx=\frac{1}{8}x^3\Big]_{15}^2=\frac{1}{8}(2)^3-\frac{1}{8}(1.5)^3=.578125.$

41. Let $X\sim$ Poisson($\mu=20$). (a) $\quad P(X\le 10)=F(10;20)=.011.$

(b) $\quad P(X>20)=1-F(20;20)=1-.559=.441.$

(c)

$$P(10\le X\le 20)=F(20;20)-F(9;20)=.559-.005=.554$$
$$P(10<X<20)=F(19;20)-F(10;20)=.470-.011=.459$$

(d)

$E(X)=\mu=20,$ so $\sigma=\sqrt{20}=4.472.$ Therefore, $P(\mu-2\sigma<X<\mu+2\sigma)=$
$P(20-8.944<X<20+8.944)\quad=P(11.056<X<28.944)=P(X\le 28)-P(X\le 11)=$
$F(28;20)-F(11;20)=.966-.021=.945.$

42. The probability that 2 heads appear is $\frac{1}{4}$, that 2 tails (no heads) appear is $\frac{1}{4}$ and that 1 head appears is $\frac{1}{2}$. Thus the probability of winning \$2 is $\frac{1}{4}$, of winning \$1 is $\frac{1}{2}$, and of losing \$5 is $\frac{1}{4}$. Hence $E=2\cdot\frac{1}{4}+1\cdot\frac{1}{2}-5\cdot\frac{1}{4}=-\frac{1}{4}=-0.25$. That is, the expected value of the game is minus \$0.25 , and so is unfavorable to the player.

43. (i) The probability of winning \$5 is $\frac{1}{4}$, of winning \$2 is $\frac{1}{2}$, and of winning \$1 is $\frac{1}{4}$; hence $E=5\cdot\frac{1}{4}+2\cdot\frac{1}{2}+1\cdot\frac{1}{4}=2.50$, that is, the expected winnings are \$2.50.

(ii) If he pays \$2.50 to play the game, then the game is fair.

44. (a) Since f is a density it integrates to 1 and so $a+b/3=1$. In addition $3/5=E[X]=\int_0^1 x\left(a+bx^2\right)dx=a/2+b/4$. Hence, $a=3/5$ and $b=6/5$.

(b) (i) Since

$$F(x)=\int_0^x e^{-x}dx=1-e^{-x}$$

if follows that

$$1/2=1-e^{-m}\qquad\text{or}\qquad m=\log(2)$$

(ii) In this case, $F(x)=x, 0\le x\le 2$; hence $m=1/2$.

45. Let X represent the annual rainfall where X follows a normal distribution with a mean of 40 and a standard deviation of 4. To find $P(X>50)$, first compute the z-score. The z-score can be found as follows:

$$\begin{aligned} z&=\frac{x-\mu}{\sigma}\\ &=\frac{50-40}{4}\\ &=2.5\end{aligned}$$

Now observe the following:

$$\begin{aligned} P(X > 50) &= 1 - P(Z \leq 2.5) \\ &= 1 - 0.9937903 \\ &\approx 0.00621 \end{aligned}$$

Now to find the probability that in 2 of the next 4 years the rainfall will exceed 50 inches, use the binomial distribution where $\boldsymbol{p} = 0.00621$ and $\boldsymbol{q} = 1 - p$, let Y be a random variable that represents years where the rainfall exceeds 50 inches and has $p = 0.00621$.

Using the probability mass function for the binomial distribution, we get the following:

$$\begin{aligned} P(Y = 2) &= \binom{4}{2}(p)^2 * (1-p)^2 \\ P(Y = 2) &= \binom{4}{2}(0.00621)^2 * (1 - 0.00621)^2 \\ &\approx 0.00023 \end{aligned}$$

So the probability that in 2 of the next 4 years that the region will receive more than 50 inches of annual rainfall is approximately 0.00023.

46. It is clear that a necessary and sufficient condition for the three segments to form a triangle is that the length of any one of the segments be less than the sum of the other two. Let x, y be the abscissas of the two points chosen at random. Then we must have either

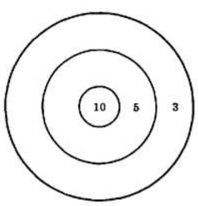
$$0 < x < \frac{1}{2} < y < 1 \text{ and } y - x < \frac{1}{2}$$

or

$$0 < y < \frac{1}{2} < x < 1 \text{ and } x - y < \frac{1}{2}.$$

This is precisely the shaded area in the Figure (See class notes). It follows that the required probability is $\frac{1}{4}$.

47. The probability of scoring $10, 5, 3$ or 0 points follows:



$$\begin{aligned} f(10) &= \frac{1}{2} \cdot \frac{\text{area of 10 points}}{\text{area of target}} = \frac{1}{2} \cdot \frac{\pi(1)^2}{\pi(5)^2} = \frac{1}{50} \\ f(5) &= \frac{1}{2} \cdot \frac{\text{area of 5 points}}{\text{area of target}} = \frac{1}{2} \cdot \frac{\pi(3)^2 - \pi(1)^2}{\pi(5)^2} = \frac{8}{50} \\ f(3) &= \frac{1}{2} \cdot \frac{\text{area of 3 points}}{\text{area of target}} = \frac{1}{2} \cdot \frac{\pi(5)^2 - \pi(3)^2}{\pi(5)^2} = \frac{16}{50} \\ f(0) &= \frac{1}{2} \end{aligned}$$

Thus $E = 10 \cdot \frac{1}{50} + 5 \cdot \frac{8}{50} + 3 \cdot \frac{16}{50} + 0 \cdot \frac{1}{2} = \frac{98}{50} = 1.96$.

48. (a) The integration of $f(x, y)$ over the whole region is

$$\begin{aligned} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dxdy &= \int_0^1\int_0^1 \frac{2}{5}(2x+3y)dxdy \\ &= \int_0^1 \left(\frac{2x^2}{5} + \frac{6xy}{5}\right)\Bigg|_{x=0}^{x=1} dy \\ &= \int_0^1 \left(\frac{2}{5} + \frac{6y}{5}\right) dy = \left(\frac{2y}{5} + \frac{3y^2}{5}\right)\Bigg|_0^1 = \frac{2}{5} + \frac{3}{5} = 1 \end{aligned}$$

(b) To calculate the probability, we use

$$
\begin{aligned}
P[(X,Y)\in A] &= P\left(0<X<\frac{1}{2}, \frac{1}{4}<Y<\frac{1}{2}\right)\\
&= \int_{1/4}^{1/2}\int_0^{1/2} \frac{2}{5}(2x+3y)dxdy\\
&= \int_{1/4}^{1/2}\left(\frac{2x^2}{5}+\frac{6xy}{5}\right)\Big|_{x=0}^{x=1/2} dy = \int_{1/4}^{1/2}\left(\frac{1}{10}+\frac{3y}{5}\right)dy\\
&= \left(\frac{y}{10}+\frac{3y^2}{10}\right)\Big|_{1/4}^{1/2}\\
&= \frac{1}{10}\left[\left(\frac{1}{2}+\frac{3}{4}\right)-\left(\frac{1}{4}+\frac{3}{16}\right)\right] = \frac{13}{160}
\end{aligned}
$$

49. (a) By definition,

$$
\begin{aligned}
g(x) &= \int_{-\infty}^{\infty} f(x,y)dy = \int_x^1 10xy^2dy\\
&= \frac{10}{3}xy^3\Big|_{y=x}^{y=1} = \frac{10}{3}x\left(1-x^3\right), 0<x<1,\\
h(y) &= \int_{-\infty}^{\infty} f(x,y)dx = \int_0^y 10xy^2dx = 5x^2y^2\big|_{x=0}^{x=y} = 5y^4, 0<y<1.
\end{aligned}
$$

Now

$$
f(y \mid x) = \frac{f(x,y)}{g(x)} = \frac{10xy^2}{\frac{10}{3}x\left(1-x^3\right)} = \frac{3y^2}{1-x^3}, 0<x<y<1.
$$

(b) Therefore,

$$
P\left(Y>\frac{1}{2}\,\middle|\, X=0.25\right) = \int_{1/2}^1 f(y \mid x=0.25)dy = \int_{1/2}^1 \frac{3y^2}{1-0.25^3}dy = \frac{8}{9}.
$$